Journal of Pure and Applied Algebra 215 (2011) 609–627

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Projective bundle ideals and Poincaré duality algebras

Larry Smith [a,*], R.E. Stong [b]

[a] AG-Invariantentheorie, Mittelweg 3, D 37133 Friedland, Federal Republic of Germany

[b] Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA

ARTICLE INFO

Article history:
Received 23 July 2008
Received in revised form 18 May 2010
Available online 20 July 2010
Communicated by A.V. Geramita

MSC: 13A15; 13A02; 16S36

ABSTRACT

The study of maximal–primary irreducible ideals in a commutative graded connected Noetherian algebra over a field is in principle equivalent to the study of the corresponding quotient algebras. Such algebras are Poincaré duality algebras. A prototype for such an algebra is the cohomology with field coefficients of a closed oriented manifold. Topological constructions on closed manifolds often lead to algebraic constructions on Poincaré duality algebras and therefore also on maximal–primary irreducible ideals. It is the purpose of this note to examine several of these and develop some of their basic properties.

The study of ideals in commutative graded[1] connected Noetherian algebras over a field is in principle equivalent to the study of the corresponding quotient algebras. If the ideal is maximal primary and irreducible the corresponding quotient algebra is an Artinian Gorenstein algebra, so maximal primary irreducible ideals are also referred to as **Gorenstein ideals**. In the graded case, Artinian Gorenstein algebras are **Poincaré duality algebras**, by which we mean the following (see e.g. [16] Section I.1): There is an integer d, called the **formal dimension** or **socle degree** of the algebra, such that the homogeneous component of degree d is 1-dimensional, all homogeneous components of degree strictly greater than d are zero, and the pairing between elements of degree i and $d-i$, for $0 \le i \le d$, given by multiplication into the homogeneous component of degree d is nonsingular. The prototype of such an algebra (apart from the cosmetic difference[2] of being graded commutative rather than commutative graded) is the cohomology with field coefficients of a closed oriented manifold.

There are several topological constructions on closed manifolds that lead[3] to algebraic constructions on Poincaré duality algebras. To name but a few, there is the projective space bundle associated to a vector bundle over a manifold (the subject of the Projective Bundle Theorem, see e.g., [42], page 62) the submanifold dual to a line bundle, and the connected sum of two manifolds. It is the purpose of this note to examine these and other constructions in a purely algebraic context, and develop some of their basic properties with an eye towards enhancing our store of examples (which are the basis for theorems in the end) of Gorenstein ideals in polynomial algebras.

The algebra appearing in the Projective Bundle Theorem lead us to define projective bundle ideals which are introduced in Section 1 and used throughout the rest of the manuscript. In Sections 2 and 3 we concentrate on the case of Gorenstein ideals. Dualizing an element in a Poincaré duality algebra (which includes dualizing a line bundle as a special case) is the subject of Section 5. One of our basic tools is the theory of *inverse systems* due to F.S. Macaulay in [13] Part IV and we assume familiarity

* Corresponding author.
E-mail address: larry@uni-math.gwdg.de (L. Smith).

[1] We adhere to the conventions of J.C. Moore as far as graded objects go. This means only *homogeneous* elements are considered unless explicitly stated otherwise. The homogeneous component of degree d of a graded object is denoted by a subscript $_d$ attached to the object.

[2] Recall that a graded algebra A is called **graded commutative** if $a' \cdot a'' = (-1)^{\deg(a')\cdot\deg(a'')} a'' \cdot a'$ for all $a', a'' \in A$, whereas it is **commutative graded** if $a' \cdot a'' = a'' \cdot a'$ for all $a', a'' \in A$. The cohomology algebra of a space is always graded commutative, and if taken with coefficients in a field of odd characteristic the square of any element of odd degree must be zero.

[3] One interprets the cohomology of the new manifold as having arisen by means of an algebraic construction on the cohomology of the original manifold(s).

0022-4049/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2010.06.011

with it as presented for example in [16] Section VI.1 (see also [14]). We also make use of basic facts concerning irreducible ideals as found in [16] Parts I and II. In Section 6 we examine a homogenization process for inverse forms motivated by the idea of splitting a trivial bundle off a vector bundle. It leads, for a given inverse form, to further inverse forms defining interesting families of maximal primary irreducible ideals in polynomial algebras.

Before going further a few words on notations, terminology and background. In the sequel $\mathbb{F}$ denotes a field and X a formal variable of degree one unless noted otherwise. If V is a finite dimensional vector space over $\mathbb{F}$ we denote by $\mathbb{F}[V]$ the polynomial algebra over $\mathbb{F}$ on V and let $\mathbb{F}[V, X]$ be the result of adjoining the formal variable X to $\mathbb{F}[V]$, put another way $\mathbb{F}[V, X] = \mathbb{F}[V][X]$. We grade $\mathbb{F}[V]$ by demanding the elements of V have degree one, so if $z_1, \ldots, z_n$ is a basis for V then $\mathbb{F}[V] = \mathbb{F}[z_1, \ldots, z_n]$, where $z_1, \ldots, z_n$ may also be thought of as formal variables of degree one, and $\mathbb{F}[V][X] = \mathbb{F}[z_1, \ldots, z_n, X]$. We will have occasion to refer to the graded maximal ideal in many different algebras, such as for example $\mathbb{F}[z_1, \ldots, z_n]$, which usually[4] would be denoted by $\overline{\mathbb{F}[z_1, \ldots, z_n]}$; this is both long and ugly. We therefore agree that, if the algebra under discussion is clear from context, then $\mathfrak{m}$ denotes its maximal ideal. Notations unexplained here are to be found in [16,24], or [34]. We often abbreviate the phrase *maximal primary* to $\mathfrak{m}$-primary.

A graded algebra generated by its homogeneous component of degree one is called **standard graded**. The vector space dimension of the homogeneous component of degree one is called the **rank** of the algebra. Standard graded Poincaré duality algebras occur in classical invariant theory as algebras of coinvariants. One of the results of R. Steinberg in [39] is that the algebra of coinvariants $\mathbb{C}[V]_G$ of a representation $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$, $V = \mathbb{C}^n$, of a finite group G over the complex field $\mathbb{C}$ is a Poincaré duality algebra if and only if $\rho(G) < \mathrm{GL}(n, \mathbb{C})$ is a reflection group. If the order $|G|$ of G is invertible in $\mathbb{F}$, the **nonmodular case**, then Steinberg's Theorem holds for finite groups over $\mathbb{F}$ (see [12] for the case of finite fields and [6] for the general case). However, over a field $\mathbb{F}$ of nonzero characteristic the ring of coinvariants $\mathbb{F}[V]_G$ of a representation $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ of a finite group may well be a Poincaré duality algebra though G contains no reflections at all (see e.g., [7,28]).

Rings of invariants over a Galois field $\mathbb{F}_q$ with $q = p^\nu$ elements support an unexpected structure derived from the Frobenius homomorphism: The operation of raising linear forms in $\mathbb{F}_q[V]$ to the q-th power preserves invariants. This operation can be used to define the **Steenrod algebra** of the Galois field (see e.g., [29]), which is denoted[5] by $\mathcal{P}^*$ and ever since [2,3] has played an important role in modular invariant theory.[6] The Steenrod algebra as such has its origins in algebraic topology (see e.g., [41]) where it was first used to impose strong restrictions on the algebras that can occur as the cohomology algebra of a space (see e.g., [40]). The existence of an unstable Steenrod algebra action on an algebra over a Galois field is a necessary condition for the algebra to be the cohomology algebra of a space. The **unstability conditions**, viz., if $\mathscr{P}^i \in \mathcal{P}^*$ is the i-th reduced Steenrod power operation then

$$\mathscr{P}^i(u) = \begin{cases} u^q & \text{if } i = \deg(u), \\ 0 & \text{if } i > \deg(u), \end{cases}$$

express both a triviality condition, viz., $\mathscr{P}^i(u) = 0$ for all $i > \deg(u)$, and, a nontriviality condition, viz., $\mathscr{P}^{\deg(u)}(u) = u^q$. It is the interplay of these two requirements that seems to endow the unstability condition with the power to yield unexpected consequences.

In contrast to the nonmodular case, in the modular case, it is not known how to characterize the representations whose rings of invariants are a polynomial algebra,[7] and hence whose coinvariant algebra would be a **complete intersection**, i.e., generated by a regular sequence,[8] nor is it known which Poincaré duality quotient algebras of $\mathbb{F}[V]$ can occur as coinvariant algebras, or if such a Poincaré duality quotient algebra would have to be a complete intersection. A necessary condition for a standard graded Artinian algebra over the Galois field $\mathbb{F}_q$ to be a coinvariant algebra is that it be an unstable algebra over the Steenrod algebra; again put in terms of ideals, that the Hilbert ideal be closed under the natural action of the Steenrod algebra on $\mathbb{F}_q[V]$. Ideals in $\mathbb{F}_q[V]$ closed under the action of the Steenrod algebra have been extensively studied, beginning perhaps with [22], where, though disguised, this closure property plays an essential part in the proof of central results (see also [25,19], and the references they contain). As of this writing a satisfactory generalization of Steinberg's Theorem to fields of nonzero characteristic is missing in the literature, references [12,6,26,27,30] contain partial results, and this open problem was one of our motivations for concentrating on the case of standard graded algebras[9] since a standard graded polynomial algebra $\mathbb{F}_q[V]$ admits a unique unstable Steenrod algebra action.

In addition to the classical topological treatment in [41], we note there is a purely algebraic introduction to the Steenrod algebra using the Frobenius homomorphism, as indicated above in [24] Chapter 11 or [29]. Here Steenrod operations are regarded as a means of encoding information hidden in the Frobenius map. The proofs in [24,29] are careful algebraic reworkings of [5,17]. For an approach to Steenrod operations using differential operators, see [45]. Last, but not least,

4 At least in topological circles.

5 For historical reasons the notation is $\mathscr{A}^*$ in the special case that $q = 2$.

6 In [8] O.E. Glenn anticipated this development in part, but in other language (see also [45]).

7 The groups involved must at least be reflection groups, see e.g., [23].

8 In [16] such ideals are called **regular ideals**.

9 As well as avoiding a great deal of arithmetic *tic–tac–toe* in the hypotheses.

Steenrod operations are also closely related to Hasse–Schmidt higher order derivations [9] and the work of O.Ore on noncommutative polynomial algebras [20]. There is a wide literature on the Steenrod algebra and its applications in topology and invariant theory; see e.g., [46] and the references therein.

For standard graded Poincaré duality algebras of the same formal dimension there is an operation called the connected sum which turns the set of isomorphism classes of such algebras into a semigroup. In [34] (see also [33] for some invariant theory arising from this) we determined all surface algebras (i.e., standard graded Poincaré duality algebras of formal dimension two) over $\mathbb{F}_2$ by, amongst other things, computing the Grothendieck group of isomorphism classes of standard graded surface algebras under the operation of connected sum. The semigroup is finitely generated and its structure mirrors faithfully the topological classification of closed surfaces. Namely it is generated by the $\mathbb{F}_2$-cohomology $\mathbb{T}$ of the torus $S^1 \times S^1$ and $\mathbb{P}$ of the projective plane $\mathbb{RP}(2)$ subject to the one relation $3[\mathbb{P}] = [\mathbb{T}] + [\mathbb{P}]$ in the Grothendieck group (where $[\quad]$ denotes the isomorphism class), so it is isomorphic to $\mathbb{Z}$ with generator $[\mathbb{P}]$.

By contrast, for Poincaré duality algebras of formal dimension strictly greater than two, the Grothendieck group fails to be finitely generated (see e.g., [34] and Section 4 below). This manuscript grew in part out of the search for constructions to provide generators for the Grothendieck group of standard graded threefolds.[10] We show in Section 4 that Poincaré duality algebras arising as quotients of $\mathbb{F}_2[z_1, \ldots, z_n, X]$ by a projective bundle ideal are, with one exception, always indecomposable (see Section 4) in the Grothendieck group. For Poincaré duality algebras of formal dimension three arising from projective bundle ideals we provide both a classification (see Proposition 4.1) and topological realizations (see the Appendix). Generally speaking, a connected sum of Poincaré duality algebras requires a rather large number of generators compared to the dimension of the homogeneous component of degree one: This is because many *cross product terms* (see the definition in Section 4 of the connected sum) must vanish. With this in mind, we were led to construct a family $P(n)$ of Poincaré duality algebras which are quotients of $\mathbb{F}[z_1, \ldots, z_n]$ by an irreducible ideal $I(n)$ that requires roughly $n!$ generators as an ideal, but, with $[P(n)]$ indecomposable in the Grothendieck group (see Section 6).

R.E. Stong died on April 10, 2008 before he had revised this manuscript; as only he could have done. The first author takes all responsibility for errors, misprints, and outright lies. The referee of this manuscript did an excellent job and the recommendations of the referee have been followed as closely as possible. The first author is extremely grateful to whoever the referee was and hopes to have done justice to the carefully thought through comments, corrections and suggestions.

1. Projective bundle ideals

If M is a closed smooth manifold and $\xi \downarrow M$ is a smooth $k+1$-dimensional vector bundle over M, then one may form the corresponding projective bundle with fibres the projective space of dimension k. The projective bundle theorem (see e.g. [42] page 62) provides a relation between the cohomology of the manifold M and the total space $\mathbb{P}(\xi \downarrow M)$ of the corresponding projective bundle. It serves as the basis of the following definition.

Definition. Let $I \subset \mathbb{F}[V, X]$ be a $\mathfrak{m}$-primary ideal and $J = I \cap \mathbb{F}[V]$. We call I a **projective bundle ideal** with **base ideal** J if $\mathbb{F}[V, X]\big/I$ is a free $\mathbb{F}[V]\big/J$-module with respect to the module structure defined by the canonical inclusion $\mathbb{F}[V]\big/J \hookrightarrow \mathbb{F}[V, X]\big/I$.

Suppose that $I \subset \mathbb{F}[V, X]$ is an $\mathfrak{m}$-primary ideal and $J = I \cap \mathbb{F}[V]$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{F} & \longleftarrow & \mathbb{F}[X] & \longleftarrow & \mathbb{F}[V, X] & \longleftarrow & \mathbb{F}[V] & \longleftarrow & \mathbb{F} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{F} & \longleftarrow & \boxed{?} & \longleftarrow & \mathbb{F}[V, X]\big/I & \longleftarrow & \mathbb{F}[V]\big/J & \longleftarrow & \mathbb{F}
\end{array}$$

where the rows are coexact[11] and the vertical maps the natural projections. Therefore the *unknown* cokernel $\boxed{?}$ must be of the form $\mathbb{F}[X]\big/(X^{k+1})$ for some integer k. The integer $k+1$ is called the **bundle dimension**. We call $\mathbb{F}[V, X]\big/I$ the **bundle algebra**, $\mathbb{F}[V]\big/J$ the **base algebra**, and $\mathbb{F}[X]\big/(X^{k+1})$ the **fibre algebra**. This is because the coexact sequence

$$(\maltese) \quad \mathbb{F} \longleftarrow \mathbb{F}[X]\big/(X^{k+1}) \longleftarrow \mathbb{F}[V, X]\big/I \longleftarrow \mathbb{F}[V]\big/J \longleftarrow \mathbb{F}$$

is an analog of the coexact sequence of cohomology algebras

$$\mathbb{F} \longleftarrow H^*(\mathbb{CP}(k); \mathbb{F}) \longleftarrow H^*(\mathbb{P}(\xi \downarrow B); \mathbb{F}) \longleftarrow H^*(B; \mathbb{F}) \longleftarrow \mathbb{F}$$

associated to a complex vector bundle $\xi \downarrow B$ of dimension $k+1$ over the base space B, where $\mathbb{P}(\xi \downarrow B)$ is the associated projective space bundle (see e.g. [42] page 62).

10 A **threefold** is a Poincaré duality algebra of formal dimension 3.

11 If $A'' \xleftarrow{f''} A \xleftarrow{f'} A'$ are maps between commutative graded connected algebras, then it is called **coexact** if $\ker f''$ is the ideal $f'(\bar{A}') \cdot A$ of A generated by the image of the augmentation ideal $\bar{A}'$ of A'. Equivalently, $f''(A) \cong \mathbb{F} \otimes_{A'} A$. The category of commutative graded connected algebras over a field has categorical images and cokernels: The image of a map $f : A' \longrightarrow A''$ being the monomorphism $\iota_f : f(A') \hookrightarrow A''$ and the cokernel the epimorphism $A'' \xrightarrow{\eta_f} \mathbb{F} \otimes'_A A''$. To say that $A'' \xleftarrow{f''} A \xleftarrow{f'} A'$ is coexact is equivalent to requiring that the natural map of the categorical cokernel of f' to the categorical image of f'' is an isomorphism. This is the categorical concept dual to exact.

Lemma 1.1. *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$ and $J = I \cap \mathbb{F}[V]$. Then $\mathbb{F}[V, X]\big/I$ is a free $\mathbb{F}[V]\big/J$-module with basis $1, X, \dots, X^k$.*

Proof. By hypothesis $\mathbb{F}[V, X]\big/I$ is a free $\mathbb{F}[V]\big/J$-module and the coexact sequence (✣) shows that it is generated by $1, X, \dots, X^k$. From the graded Nakayama Lemma (see e.g. [24] Proposition 5.1.3) it therefore follows that $1, X, \dots, X^\ell$ is a basis where ℓ is the smallest integer such that $X^{\ell+1}$ can be written as an $\mathbb{F}[V]\big/J$-linear combination of $1, X, \dots, X^\ell$. By definition ℓ is just k. □

This lemma means that for a projective bundle ideal $I \subset \mathbb{F}[V, X]$ of bundle dimension $k+1$ one may write

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1} \in \mathbb{F}[V, X]\big/I$$

where $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{F}[V]\big/J$ and $\deg(\alpha_j) = j$ for $j = 1, \dots, k+1$. We call this the **bundle relation**. Its coefficients are algebraic analogs of characteristic classes (see e.g., [42]). If we choose $h_1, \dots, h_{k+1} \in \mathbb{F}[V]$ lifting $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{F}[V]\big/J$ respectively, then the form

$$h(X) = X^{k+1} + h_1 X^k + \cdots + h_k X + h_{k+1}$$

belongs to I. This follows directly from the bundle relation. We call $h(X)$ a **homogenizing form** or **polynomial** for I.

Lemma 1.2. *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$, with base ideal $J = I \cap \mathbb{F}[V]$, and bundle relation*

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1},$$

where $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{F}[V]\big/J$. Then the kernel of the natural map

$$\big(\mathbb{F}[V]\big/J\big)[X] \xrightarrow{\varphi} \mathbb{F}[V, X]\big/I$$

is the principal ideal generated by

$$\alpha(X) = X^{k+1} + \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1}.$$

Proof. We begin by assembling some obvious facts.

(1) $\alpha(X) \in \ker(\varphi)$.
(2) No nonzero element $f(X) = f_0 + f_1 X + \cdots + f_m X^m$ with $m < k+1$ belongs to $\ker(\varphi)$ since $1, X, \dots, X^k$ are $\mathbb{F}[V]\big/J$-linearly independent in $\mathbb{F}[V, X]\big/I$ (by Lemma 1.1).
(3) Any nonzero element in $(\mathbb{F}[V]\big/J)[X]$ of degree $k+1$ in X belonging to the kernel of φ is a scalar multiple of $\alpha(X)$.

Hence we may use induction on the degree in X of an element $f(X) \in (\mathbb{F}[V]\big/J)[X]$ belonging to $\ker(\varphi)$ to show it belongs to $(\alpha(X))$. Choose an element $0 \neq f(X) = f_0 + f_1 X + \cdots + f_m X^m$ belonging to $\ker(\varphi)$. Then $m \geq k+1$. Hence $\alpha(X) \cdot X^{m-(k+1)}$ is a polynomial of degree m in X and moreover $f(X) - f_m \cdot \alpha(X) \cdot X^{m-(k+1)}$ belongs to the kernel of φ and has degree in X strictly less than m, so by the induction hypothesis is also in $(\alpha(X))$ and the result follows. □

Lemma 1.3. *Suppose that $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$, $J = I \cap \mathbb{F}[V]$ is the base ideal, and*

$$-X^{k+1} = \alpha_1 X^k + \cdots + \alpha_k X + \alpha_{k+1} \in \mathbb{F}[V, X]\big/I$$

is the bundle relation. Let

$$h(X) = X^{k+1} + h_1 X^k + \cdots + h_k X + h_{k+1} \in \mathbb{F}[V]$$

be a homogenizing form. Then $I = (J, h(X)) \subset \mathbb{F}[V, X]$.

Proof. Pass down from $\mathbb{F}[V, X]$ to $\big(\mathbb{F}[V]\big/J\big)[X]$ and note that the kernel of the natural map $\varphi : \big(\mathbb{F}[V]\big/J\big)[X] \longrightarrow \mathbb{F}[V, X]\big/I$ is the ideal I/J of $\big(\mathbb{F}[V]\big/J\big)[X]$. The result then follows from Lemma 1.2. □

The topological model for the following elementary example is a 2-plane bundle $\xi \downarrow \mathbb{RP}(n-1)$ with total Stiefel–Whitney class $1+z \in H^*(\mathbb{RP}(n-1); \mathbb{F}_2) = \mathbb{F}_2[z]\big/(z^n)$. Although this example is quite simple, it exhibits some complex phenomena.

Example 1. The algebra in question is a quotient of $\mathbb{F}_2[z, X]$ by a $\mathfrak{m}$-primary ideal I arising as follows. As base algebra we choose $\mathbb{F}_2[z]/(z^n)$, so the base ideal is $J = (z^n)$, and as homogenizing form $X^2 + zX \in \mathbb{F}[z, X]$. Then $I = (z^n, X^2 + zX)$. The corresponding quotient algebra $\mathbb{F}_2[z, X]\big/(z^n, X^2 + zX)$ turns out to be isomorphic to $\mathbb{F}_2[x]\big/(x^n)\#\mathbb{F}_2[y]\big/(y^n)$ where # denotes the connected sum operation of Poincaré duality algebras introduced in Section I.5 of [16]. The precise definition of # is also at the beginning of Section 4 of this manuscript.

One way to see this is to consider the isomorphism φ of $\mathbb{F}_2[z, X]$ with $\mathbb{F}_2[x, y]$ induced by sending z to $x + y$ and X to y. Then

$$\varphi(X(X + z)) = xy$$
$$\varphi(z^n) = (x + y)^n \equiv x^n + y^n \bmod(xy),$$

so φ maps $(z^n, X^2 + zX)$ isomorphically onto the ideal $(x^n + y^n, xy)$. An alternative version of this proof is to be found in Section 2 Example 1. As is shown in Section 4, this is the only projective bundle ideal with ground field $\mathbb{F}_2$ which produces a Poincaré duality algebra that is a connected sum.

Remark. In the case where the ground field is a finite field $\mathbb{F}_q$ with q elements and $I \subset \mathbb{F}_q[V, X]$ is a projective bundle ideal with base ideal $J = I \cap \mathbb{F}_q[V]$ then Lemma 1.3 shows I is closed under the action of the Steenrod algebra if and only if J is closed under the action of the Steenrod algebra and one can choose a homogenizing form $h(X)$ so that its image $\alpha(X)$ in $\left(\mathbb{F}[V]\big/J\right)[X]$ is a Thom class.[12] The latter certainly is the case if one can choose $h(X)$ to be a product of linear forms in $\mathbb{F}_q[V, X]$ (see e.g., [1]).

Lemma 1.4. *Suppose that $I \subset \mathbb{F}[X, y, z]$ is a projective bundle ideal. Then I is a complete intersection.*

Proof. Let $J = I \cap \mathbb{F}[y, z]$ be the bundle ideal. By a result of W. Vasconcelos [43] (see also [32] for an elementary proof of this result) J is generated by a regular sequence, say f', f''. So by Lemma 1.3 I is generated by f', f', h which must then be a regular sequence since I is $\mathfrak{m}$-primary. □

2. Projective bundle ideals with an irreducible base ideal

We are particularly interested in projective bundle ideals which are irreducible. The next pair of results show that the assumption a projective bundle ideal is irreducible is equivalent to assuming that its base ideal is irreducible. For related results see [35].

Definition. The **Poincaré series** of a graded vector space X whose homogeneous components X_i for $i \in \mathbb{N}_0$ are finite dimensional (such an X is said to be of **finite type**) is the formal series $P(X, t) = \sum_{i=0}^{\infty} \dim_{\mathbb{F}}(X_i)$ in $\mathbb{N}_0[t]$. The terminology **Hilbert series** or **Hilbert function** of X is also in common use for $P(X, t)$. If $X_i = 0$ for i large then the Poincaré series is in fact a polynomial which we call the **Poincaré polynomial**[13] of X and is sometimes specified by writing out the coefficients enclosed in parentheses, viz., $(\dim_F(X_0), \dim_{\mathbb{F}}(X_1), \ldots, \dim_{\mathbb{F}}(X_\ell))$ if $X_i = 0$ for $i > \ell$.

Lemma 2.1. *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal with base ideal $J = I \cap F[V]$ that is irreducible in $\mathbb{F}[V]$. Then $I \subset \mathbb{F}[V, X]$ is irreducible and $\mathbb{F}[V, X]\big/I$ is a Poincaré duality algebra of formal dimension $d + k$, where d is the formal dimension of the base algebra $\mathbb{F}[V]\big/J$ and $k + 1$ is the bundle dimension.*

Proof. By [16] Lemma I.1.3 and Proposition I.1.5, an $\mathfrak{m}$-primary ideal in a polynomial algebra is irreducible if and only if the corresponding quotient algebra is a Poincaré duality algebra. Thus $\mathbb{F}[V]\big/J$ is a Poincaré duality algebra and we need to show that $\mathbb{F}[V, X]\big/I$ is also. To this end note that the fundamental coexact sequence (✣) of Section 1 shows that the Poincaré polynomial of $\mathbb{F}[V, X]\big/I$ is of degree $d + k$ and that the homogeneous component in this degree of $\mathbb{F}[V, X]\big/I$ is 1-dimensional. Choose a fundamental class $u \in \mathbb{F}[V]\big/J$. It has degree d. We will show that uX^k serves as a fundamental class for $\mathbb{F}[V, X]\big/I$.

Let $f \in \mathbb{F}[V, X]$ be a nonzero element and write $f = f_0 + f_1X + \cdots + f_kX^k$ where $f_0, \ldots, f_k \in \mathbb{F}[V]\big/J$. Let ℓ be the largest integer such that $f_\ell \neq 0$, so in point of fact $f = f_0 + f_1X + \cdots + f_\ell X^\ell$. By homogeneity $\deg(f_\ell) < \deg(f_{\ell-1}) < \cdots < \deg(f_0) = \deg(f)$. Let $f_\ell^\vee \in \mathbb{F}[V]\big/J$ be a Poincaré dual for f_ℓ in $\mathbb{F}[V]\big/J$. Then $\deg(f_\ell^\vee) = d - \deg(f_\ell)$ so, for $i = 1, \ldots, \ell$, we have

$$\deg(f_\ell^\vee f_{\ell-i}) = \deg(f_\ell^\vee) + \deg(f_{\ell-i}) = d - \deg(f_\ell) + \deg(f_{\ell-i}) > d,$$

and hence $f_\ell^\vee f_{\ell-i} = 0 \in \mathbb{F}[V]\big/J$, $i = 1, \ldots, \ell$, for degree reasons. Therefore

$$f_\ell^\vee X^{k-\ell} \cdot f = f_\ell^\vee f_0 X^{k-\ell} + f_\ell^\vee f_1 X^{k-\ell+1} + \cdots + f_\ell^\vee f_\ell X^k = uX^k$$

showing that uX^k serves as a fundamental class for $\mathbb{F}[V, X]\big/I$ and completing the proof. □

Lemma 2.2. *If $I \subset \mathbb{F}[V, X]$ is an irreducible projective bundle ideal with a base ideal $J = I \cap F[V]$, then $J \subset \mathbb{F}[V]$ is irreducible. The Poincaré polynomials of the terms of the fundamental coexact sequence are related by the formula*

$$P(\mathbb{F}[V, X]\big/I, t) = P(\mathbb{F}[V]\big/J, t) \cdot P(\mathbb{F}[X]\big/(X^{k+1}), t).$$

12 One says that an element $a \in A$ of unstable algebra A over the Steenrod algebra $\mathcal{P}^*$ is a **Thom class** if and only if the principal ideal it generates is closed under the action of the Steenrod algebra.

13 The term **Hilbert polynomial** is **not** used since it has yet another meaning.

Proof. Let $k+1$ be the bundle dimension of I. Consider the fundamental coexact sequence $(\maltese)$ of Section 1. Since $\mathbb{F}[V, X]\big/I$ is a free $\mathbb{F}[V]\big/J$-module this sequence splits as a sequence of $\mathbb{F}[V]\big/J$-modules. Choose a splitting

$$s : \mathbb{F}[X]\big/(X^{k+1}) \longrightarrow \mathbb{F}[V, X]\big/I$$

as $\mathbb{F}$-vector spaces of the quotient map $\mathbb{F}[V, X]\big/I \longrightarrow \mathbb{F}[X]\big/(X^{k+1})$. Since $\mathbb{F}[V, X]\big/I$ is free as an $\mathbb{F}[V]\big/J$-module with $\mathbb{F} \otimes_{\mathbb{F}[V]\big/J} \mathbb{F}[V, X]\big/I$ as indecomposable elements, the map

$$\mu : \big(\mathbb{F}[V]\big/J\big) \otimes \big(\mathbb{F}[X]\big/(X^{k+1})\big) \longrightarrow \mathbb{F}[V, X],$$

defined by $\mu(a, f) = a \cdot f$, is an isomorphism of $\mathbb{F}[V]\big/J$-modules. So for the Poincaré polynomials of the terms of the fundamental coexact sequence one has the product formula

$$P(\mathbb{F}[V, X]\big/I, t) = P(\mathbb{F}[V]\big/J, t) \cdot P(\mathbb{F}[X]\big/(X^{k+1}), t).$$

From this it follows that $P\big(\mathbb{F}[V]\big/J, t\big)$ has degree d, where $k + d = f\text{-dim}(\mathbb{F}[V, X]\big/I)$, and moreover $\mathbb{F}[V]\big/J$ is one dimensional over $\mathbb{F}$ in homogeneous degree d. Choose a nonzero element $u \in \mathbb{F}[V]\big/J$ of degree d. To show $J \subset \mathbb{F}[V]$ is irreducible we invoke [16] Lemma I.1.3 and show instead that $\mathbb{F}[V]\big/J$ is a Poincaré duality algebra with fundamental class u.

To this end suppose that $0 \neq a \in \mathbb{F}[V]\big/J$. Then in $\mathbb{F}[V, X]\big/I$ the element a has a Poincaré dual, say

$$a^\vee = a_0^\vee + a_1^\vee X + \cdots + a_k^\vee X^k.$$

By homogeneity

$$\deg(a_0^\vee) > \deg(a_1^\vee) > \cdots > \deg(a_k^\vee) = d,$$

so $a_0^\vee = \cdots = a_{k-1}^\vee = 0$ since $\mathbb{F}[V]\big/J$ is identically zero in homogeneous degrees exceeding d. Hence $uX^k = aa^\vee = aa_k^\vee X^k$ and therefore $u = aa_k^\vee$ since $1, X, \ldots, X^k$ is a basis for $\mathbb{F}[V, X]\big/I$ as an $\mathbb{F}[V]\big/J$-module by the graded Nakayama Lemma ([24] Proposition 5.1.3). Hence $a_k^\vee$ serves as a Poincaré dual to a in $\mathbb{F}[V]\big/J$ with respect to u. So u is a fundamental class for $\mathbb{F}[V]\big/J$, making it a Poincaré duality algebra. $\square$

Remark. The product formula in Lemma 2.2 is motivated by one of the proofs of the Projective Bundle Theorem of algebraic topology which shows that the Serre spectral sequence of the fibration $\mathbb{CP}(n-1) \hookrightarrow E \downarrow X$ obtained by projectivizing a complex vector bundle $\xi \downarrow X$ over X collapses. This product formula implies that the first difference sequence of the Poincaré series of the quotient of $\mathbb{F}[V, X]$ by a projective bundle ideal is what has been called a Gorenstein sequence in the literature (see e.g., [38] Section 4) so could itself appear as the coefficients of the Poincaré polynomial of a Poincaré duality algebra.

Proposition 2.3. *If $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal with base ideal $J = I \cap F[V]$, then I is irreducible in $\mathbb{F}[V, X]$ if and only if J is irreducible in $\mathbb{F}[V]$.*

Proof. Combine Lemmas 2.1 and 2.2. $\square$

Choose a basis $z_1, \ldots, z_n$ for V and let $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ denote the algebra of inverse polynomials (see [16] Section VI.1). Introduce as in [16] Part VI Section 1. the notation $\cap$ for the action of a polynomial algebra on its companion algebra of inverse polynomials.[14] If $I \subset \mathbb{F}[V, X]$ is a projective bundle ideal with an irreducible base ideal $J = I \cap \mathbb{F}[V]$, then J is also irreducible. So both I, respectively J, have Macaulay inverses[15] ([16] Section VI.2) $\theta_I \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}]$, respectively $\theta_J \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$. In the theorem that follows we show that θ_I arises from θ_J by means of a homogenization process. We first require a lemma.

Lemma 2.4. *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal of bundle dimension $k+1$ with irreducible base ideal $J = I \cap \mathbb{F}[V]$. Set $d = f\text{-dim}(\mathbb{F}[V]\big/J)$ and let $h(X) \in \mathbb{F}[V, X]$ be a homogenizing form for I. Then $X^{k+d+1} \in I$ and there exists a form $\overline{h}(X)$ in $\mathbb{F}[V, X]$ of degree d in X, say*

$$\overline{h}(X) = X^d + \overline{h}_1 X^{d-1} + \cdots + \overline{h_d},$$

whose coefficients belong to $\mathbb{F}[V]$ and are well defined modulo the bundle ideal J, such that

$$\overline{h}(X)h(X) = X^{k+1+d} \in \big(\mathbb{F}[V]\big/J\big)[X].$$

14 This action is often referred to as the **contraction pairing** by analogy with the terminology of classical tensor calculus. The context should make clear if $\cap$ is the contraction pairing or intersection of sets.

15 One needs to choose a basis $z_1, \ldots, z_n$ for V to define the $\mathbb{F}[z_1, \ldots, z_n]$-module structure on $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ needed to apply Macaulay's theory. We always assume this has been done in a context involving Macaulay inverses.

Proof. It follows directly from the hypotheses that the Poincaré series for $\mathbb{F}[V, X]\big/I$ is a polynomial of degree $d + k$, and hence $X^{k+d+1} \in I$ for degree reasons. By Lemma 1.3 $I = (J, h(X))$, so passing down to $\big(\mathbb{F}[V]\big/J\big)[X]$ one sees that X^{k+d+1} being zero in $\mathbb{F}[V, X]\big/I$ implies that in $\big(\mathbb{F}[V]\big/J\big)[X]$ it belongs to the principal ideal generated by $h(X)$ and the result follows. $\square$

A form $\overline{h}(X) \in \mathbb{F}[V, X]$ with the properties of Lemma 2.4 is called a **dual homogenizing form** or **polynomial** for the projective bundle ideal $I \subset \mathbb{F}[V, X]$.

Theorem 2.5. *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal of bundle dimension $k + 1$ with irreducible base ideal $J = I \cap \mathbb{F}[V]$. If $\overline{h}(X) = X^d + \overline{h}_1 X^{d-1} + \cdots + \overline{h}_d \in \big(\mathbb{F}[V]\big/J\big)[X]$ is a dual homogenizing form for I (so $d = f\text{-}\dim(\mathbb{F}[V]\big/J)$) and $\theta_J \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ is a Macaulay inverse for J, then (using $\cap$ to denote the action of polynomials on inverse polynomials)*

$$\theta_I = \overline{h}(X) \cap (\theta_J \cdot X^{-(d+k)}) = \theta_J X^{-k} + (\overline{h}_1 \cap \theta_J)X^{-(k+1)} + \cdots + (\overline{h}_d \cap \theta_J)X^{-(k+d)}$$

is a Macaulay inverse for I.

Proof. We have the inclusion of ideals $(J, X^{k+d+1}) \subseteq I = (J, h(X))$. The ideal (J, X^{k+d+1}) is irreducible in $\mathbb{F}[V, X]$ since

$$\mathbb{F}[V, X]\big/(J, X^{k+d+1}) \cong \big(\mathbb{F}[V]\big/J\big) \otimes \big(\mathbb{F}[X]\big/(X^{k+d+1}\big)$$

is a tensor product of Poincaré duality algebras (see e.g. [16] Lemma I.1.3) and hence a Poincaré duality algebra in its own right. Moreover, this tensor product splitting shows that the Macaulay inverse for the ideal (J, X^{k+d+1}) is $\theta_J \cdot X^{-(k+d)} \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}] \otimes \mathbb{F}[X^{-1}] = \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}]$. We next apply the $K \subset L$ paradigm (see [16] Theorem II.5.1) to the pair of ideals $(J, X^{k+d+1}) \subseteq I$. To do so we require a transition element for I over (J, X^{k+d+1}) (see [10] and [16] Part I for basic facts about transition elements). To this end note that by standard properties of the $(- : -)$ construction one has

$$((J, X^{k+d+1}) : I) = ((J, X^{k+d+1}) : (J, h(X))) = ((J, X^{k+d+1}) : (h(X))) = (\overline{h}(X)) + (J, X^{k+d+1})$$

by the definition of $\overline{h}(X)$. So $\overline{h}(x)$ serves as a transition element for I over (J, X^{k+d+1}) and applying [16] Theorem II.5.1 yields the desired conclusion. $\square$

Example 1. Consider the following modification[16] of Section 1 Example 1. In this situation $J = (z^n) \subset \mathbb{F}[z]$, corresponding to the base algebra being $\mathbb{F}[z]\big/(z^n)$, which has formal dimension $n-1$. The homogenizing quadratic form is $h(X) = X^2 - zX \in \mathbb{F}[z, X]$, which means the bundle dimension is 2. The corresponding dual homogenizing form (see Lemma 2.4) is then

$$\overline{h}(X) = X^{n-1} + zX^{n-2} + \cdots + z^{n-1},$$

as simple multiplication verifies. The inverse form $\theta_j = z^{1-n} \in \mathbb{F}[z^{-1}]$ is a Macaulay inverse for the ideal $I = (z^n) \subset \mathbb{F}[z]$ defining the base algebra. As a consequence of Theorem 2.5 the bundle ideal $(z^n, X^2 - zX)$ is therefore defined by the inverse form

$$\theta_I = \overline{h}(X) \cap z^{1-n}X^{-(n-1)} = z^{-n} + z^{-(n-1)}X^{-1} + \cdots + z^{-1}X^{-(n-1)} \in \mathbb{F}[z^{-1}, X].$$

If we replace z by y and X by x then this inverse form becomes

$$\theta = x^{-n} + x^{-(n-1)}y^{-1} + \cdots + x^{-1}y^{-(n-1)} \in \mathbb{F}[z^{-1}, X] = \mathbb{F}[x^{-1}, y^{-1}]$$

and is in the same $\mathrm{GL}(2, \mathbb{F})$-orbit as the inverse form $x^{-n} - y^{-n} \in \mathbb{F}[x^{-1}, y^{-1}]$ since the transvection $x \rightsquigarrow x, y \rightsquigarrow x - y$ in $\mathrm{GL}(2, \mathbb{F})$ sends the inverse form $x^{-n} - y^{-n}$ to the inverse form $x^{-n} - x^{-(n-1)}y^{-1} + \cdots + x^{-1}y^{-(n-1)}$. The inverse form $x^{-n} - y^{-n}$ is the Macaulay inverse for the ideal $(xy, x^n - y^n)$. To verify this, one first writes down the catalecticant matrix (see [16] Section VI.2) $cat_\theta(1, n-1)$ from which one sees that $xy, x^n - y^n \in I(\theta)$. Then one notes that the induced map

$$\mathbb{F}[x, y]\big/(xy, x^n - y^n) \longrightarrow \mathbb{F}[x, y]\big/I(\theta)$$

has degree one, since both these Poincaré duality algebras have x^n as a fundamental class. Finally one applies [16] Corollary I.2.4 to conclude this map is an isomorphism, so the inclusion $(xy, x^n - y^n) \subseteq I(\theta)$ must be an equality. This justifies the remark made in Section 2 Example 1 concerning the structure of the quotient algebra $\mathbb{F}[z, X]\big/J$.

[16] We are grateful to the referee for pointing out that two sign changes make this example characteristic free.

3. Bundling an $\mathfrak{m}$-primary irreducible ideal

We next show that a converse of Theorem 2.5 holds, allowing the construction of a family of $\mathfrak{m}$-primary irreducible ideals in $\mathbb{F}[V, X]$ indexed by monic polynomials of strictly positive degree in $\mathbb{F}[V][X]$ from a single $\mathfrak{m}$-primary ideal J in $\mathbb{F}[V]$. Suppose we are given an $\mathfrak{m}$-primary irreducible ideal $J \subset \mathbb{F}[V]$ and a monic polynomial in $\mathbb{F}[V, X]$ of degree $k+1$ in X. Write

$$h(X) = X^{k+1} + h_1X^k + \cdots + h_kX + h_{k+1},$$

where the elements $h_1, \ldots, h_{k+1} \in \mathbb{F}[V]$ have strictly positive degrees in $\mathbb{F}[V]$, so belong to the augmentation ideal of $\mathbb{F}[V]$. Guided by Lemma 1.3 we consider the ideal $I = (J, h(X)) \subset \mathbb{F}[V, X]$. If d is the degree of the Poincaré polynomial of $\mathbb{F}[V]\big/J$, then $1, X, \ldots, X^d$ generate $\mathbb{F}[V, X]\big/I$ as an $\mathbb{F}[V]\big/J$-module, so one sees that the Poincaré series of $\mathbb{F}[V, X]\big/I$ is a polynomial of degree $k + d$. Therefore we have proven the following lemma.

Lemma 3.1. *If $J \subset \mathbb{F}[V]$ is an $\mathfrak{m}$-primary irreducible ideal and $h(X) \in \mathbb{F}[V, X]$ is a monic polynomial in X of strictly positive degree $k + 1$, then $I = (J, h(X)) \subset \mathbb{F}[V, X]$ is an $\mathfrak{m}$-primary ideal, the Poincaré polynomial of $\mathbb{F}[V, X]\big/I$ has degree $k + d$, where d is the degree of the Poincaré polynomial of $\mathbb{F}[V]\big/J$, and $X^{k+d+1} \in I$.* $\square$

Continuing in this vein one has:

Lemma 3.2. *Let A be a commutative graded connected algebra over a field and $A[X]$ the polynomial algebra over A in the variable X of some strictly positive degree. If $\alpha(X) \in A[X]$ is a monic polynomial of degree m in X then $A[X]\big/(\alpha(X))$ is a free A-module with basis $1, X, \ldots, X^{m-1}$.*

Proof. One notes that homogeneity requires that the coefficients of $\alpha(X)$ apart from the coefficient of X^m belong to the augmentation ideal of A. Hence $1 \otimes (X^i \cdot \alpha(X)) \equiv 1 \otimes X^{m+i}$ in $\mathbb{F} \otimes_A A[X]$, for $i = 0, 1, \ldots,$ so

$$1, X, \ldots, X^{m-1}, \alpha(X), X \cdot \alpha(X), \ldots,$$

project to an $\mathbb{F}$-vector space basis for the module of A-indecomposables $\mathbb{F} \otimes_A A[X]$. Since $A[X]$ is a free A-module they therefore are an A-basis for $A[X]$ by the graded Nakayama Lemma (see e.g. [24] Proposition 5.1.3). Since $\alpha(X), X \cdot \alpha(X), \ldots,$ span the ideal $(\alpha(X))$ as an A-module it follows that the inclusion $(\alpha(X)) \hookrightarrow A$ splits as a map of A-modules. Hence $A[X]\big/(\alpha(X))$ is also free as an A-module and $1, X, \ldots, X^{m-1}$ project to a basis for it. $\square$

Lemma 3.2 says in particular: For any ideal $J \subset \mathbb{F}[V]$ and any monic polynomial $h(X) \in \mathbb{F}[V, X]$ in X the quotient algebra $\mathbb{F}[V, X]\big/(J, h(X))$ is free as an $\mathbb{F}[V]\big/J$-module. To see this simply set $A = \mathbb{F}[V]\big/J$ in the lemma.

For an $\mathfrak{m}$-primary irreducible ideal $J \subset \mathbb{F}[V]$ denote by d_J, or d if J is clear from context, the degree of the Poincaré polynomial of $\mathbb{F}[V]\big/J$, i.e., $d = f\text{-dim}(\mathbb{F}[V]\big/J)$ is the formal dimension (or socle degree) of the Poincaré duality quotient algebra $\mathbb{F}[V]\big/J$. Let $h(X) \in \mathbb{F}[V, X]$ be a monic polynomial in X of strictly positive degree $k+1$, so that $h(X) = X^{k+1} + h_1X^k + \cdots h_kX + h_{k+1}$, where $h_i \in \mathbb{F}[V]$ has degree i for $i = 1, \ldots, k+1$. The maximal ideal of $\mathbb{F}[V]\big/J$ is nilpotent, so the inhomogeneous element $1 + h_1 + \cdots + h_k + h_{k+1}$ has a formal inverse in the ungraded algebra[17] $\mathrm{Tot}(\mathbb{F}[V]\big/J)$. Say $1 + \overline{h}_1 + \cdots + \overline{h}_d$ is such an inverse, where $\overline{h}_j \in \mathbb{F}[V]\big/J$ has degree j for $j = 1, \ldots, d$, so in $\mathrm{Tot}\big(\mathbb{F}[V]/J\big)$

$$(✣) \quad (1 + h_1 + \cdots h_{k+1})(1 + \overline{h}_1 + \cdots + \overline{h}_d) = 1.$$

Introduce the polynomial

$$\overline{h}(X) = X^d + \overline{h}_1X^{d-1} + \cdots + \overline{h}_d \in \big(\mathbb{F}[V]\big/J\big)[X].$$

Then $(✣)$ shows that

$$(✠) \quad h(X) \cdot \overline{h}(X) = X^{k+d+1} \in \big(\mathbb{F}[V]\big/J\big)[X].$$

This leads to the following result.

Lemma 3.3. *With the notations preceding one has $(J, X^{k+d+1}) \subseteq (J, h(X))$ and*

$$\big((J, X^{k+d+1}) : (J, h(X))\big) = \big(\overline{h}(X)\big) + (J, X^{k+d+1}).$$

Proof. The first statement follows from Lemma 3.1. Pass down to the quotient algebra $B = \mathbb{F}[V, X]\big/(J, X^{k+d+1})$, then the second statement becomes equivalent to showing that $\mathrm{Ann}_B(h(X)) = (\overline{h}(X))$. The Eq. $(✠)$ shows that $(\overline{h}(X)) \subseteq \mathrm{Ann}_B(h(X))$, so it remains to prove the reverse inclusion.

17 $\mathrm{Tot}(\Xi)$ for any graded vector space Ξ is the direct product of its homogeneous components.

Suppose that $0 \neq f(X) = f_0X^m + \cdots + f_m \in \mathrm{Ann}_B(h(X))$, where $f_0, \ldots, f_m \in A = \mathbb{F}[V]\big/(J)$. Without loss of generality we may assume that $f_0 \neq 0$ so $f(X)$ is of degree m in X. But

$$0 = f(X) \cdot h(X) = f_0X^{k+m+1} + \text{ terms of lower degree in } X,$$

so $k + m + 1 \geq k + d + 1$ since B is a free A-module with basis $1, X, \ldots, X^{k+d}$. If $m = d$, then $f(X)$ and $\overline{h}(X)$ both have degree d, so $f(X) - f_0\overline{h}(X) \in \mathrm{Ann}_B(h(X))$ and this element has degree at most $d-1$ in X. By what was just shown this means $f(X) - f_0\overline{h}(X)$ is identically zero, so $f(X) = f_0\overline{h}(X)$ and $f(X)$ belongs to the principal ideal generated by $\overline{h}(X)$. Hence we may proceed inductively and assume that for all $\bar{m} < m$, if $\bar{f}(X) \in B$ has degree $\bar{m}$ in X and annihilates $h(X)$ then $\bar{f}(X) \in ((\overline{h}(X))$. If $f(X) \in \mathrm{Ann}_B(h(X))$ has degree m in X, then writing $f(X)$ as above one sees that $f(X) - f_0\overline{h}(X)$ has degree at most $m-1$ in X and it too annihilates $h(X)$. By the inductive assumption, $f(X) - f_0\overline{h}(X)$ belongs to $(\overline{h}(X))$ and, hence, so does $f(X)$, completing the inductive step. □

Theorem 3.4. *Let $J \subset \mathbb{F}[V]$ be an $\mathfrak{m}$-primary ideal and $h(X) \in \mathbb{F}[V, X]$ a monic polynomial in X of strictly positive degree $k+1$. Then $I = (J, h(X)) \subset \mathbb{F}[V, X]$ is a projective bundle ideal of bundle dimension $k+1$ with base ideal J. If J is irreducible in $\mathbb{F}[V]$ then $I = (J, h(X))$ is irreducible in $\mathbb{F}[V, X]$.*

Proof. The ideal I is $\mathfrak{m}$-primary by Lemma 3.1 and $\mathbb{F}[V, X]\big/I$ is free as an $\mathbb{F}[V]\big/J$-module by Lemma 3.2. Since $J = I \cap \mathbb{F}[V]$ it follows from Lemma 1.3 that I is a projective bundle ideal with base ideal J and bundle dimension $k+1$. If J is irreducible in $\mathbb{F}[V]$, then (J, X^{k+d+1}) is irreducible in $\mathbb{F}[V, X]$, where d is the formal dimension of $\mathbb{F}[V]\big/J$. To see this one notes that

$$\mathbb{F}[V, X]\big/(J, X^{k+d+1}) \cong \big(\mathbb{F}[V]/J\big) \otimes \big(\mathbb{F}[X]\big/(X^{k+d+1})\big)$$

is a Poincaré duality algebra by [16] Proposition I.1.5 and then applies [16] Lemma I.1.3 to conclude that $(J, X^{k+d+1}) \subset \mathbb{F}[V, X]$ is an irreducible ideal. By Lemma 3.1 $(J, X^{k+d+1}) \subseteq (J, h(X))$ and by Lemma 3.3 $\big((J, X^{k+d+1}) : (J, h(X))\big) = \big(\overline{h}(X)\big) + (J, X^{k+d+1})$, so I is irreducible by [16] Theorem I.2.1. Alternatively, as pointed out by the referee, since J is irreducible $\mathbb{F}[V]\big/J$ is Gorenstein, hence so is $(\mathbb{F}[V]\big/J)[X]$. The polynomial $h(X)$ is a regular element in $(\mathbb{F}[V]\big/J)[X]$, so $\mathbb{F}[V, X]\big/(J, h(X)) = (\mathbb{F}[V]\big/J)[X]\big/(h(X))$ is Gorenstein by [4] Proposition 3.1.19 and, therefore, $(J, h(X))$ is irreducible in $\mathbb{F}[V, X]$. □

Remark. In the situation of Lemma 3.3 one has shown in the quotient algebra

$$B = \mathbb{F}[V, X]\big/(J, X^{k+d+1}) = \Big(\mathbb{F}[V]\big/J\Big) \otimes \Big(\mathbb{F}[X]\big/(X^{k+d+1})\Big)$$

that

$$(0 \underset{B}{:} h(X)) = (\overline{h}(X)),$$

or what is the same thing, that

$$\mathrm{Ann}_B(h(X)) = (\overline{h}(X)).$$

The algebra B is Noetherian and $(0) \subset B$ is an irreducible ideal since B is a Poincaré duality algebra. Therefore by Emmy Noether's Involution Theorem (see [16] Section I.1.2) one also has

$$(0 \underset{B}{:} \overline{h}(X)) = (h(X))$$

which can be rephrased as

$$\mathrm{Ann}_B(\overline{h}(X)) = (h(X)).$$

Thus in the algebra B the images of $h(X)$ and $\overline{h}(X)$ are mutual annihilators of each other. So by [16] Corollary I.2.3 one has the following conclusions.

(i) $B/(\overline{h}(X))$ is a Poincaré duality algebra of formal dimension

$$f\text{-}\dim(B) - \deg(\overline{h}(X)) = d + k + d - (k+1) = 2d - 1.$$

(ii) $B/(\overline{h}(X))$ is a Poincaré duality algebra of formal dimension

$$f\text{-}\dim(B) - \deg(\overline{h}(X)) = d + k + d - d = k + d,$$

(as of course it should be since $B/(h(X)) \cong \mathbb{F}[V, X]\big/(J, h(X))$).

If $X^{k+d+1} \in (J, \overline{h}(X)) \subset \mathbb{F}[V, X]$, then by Theorem 3.4 the relation of Lemma 3.3 would also hold with the roles of $h(X)$ and $\overline{h}(X)$ switched, viz.,

$$\big((J, X^{k+d+1}) : (J, \overline{h}(X))\big) = \big(h(X)\big) + (J, X^{k+d+1}).$$

This would give us another projective bundle ideal $(J, \overline{h}(X)) \subset \mathbb{F}[V, X]$. The condition $X^{k+d+1} \in (J, \overline{h}(X))$ holds if for example

$$k + d + 1 > f\text{-}\dim(\mathbb{F}[V, X]\big/(J, \overline{h}(X)) = 2d - 1$$

i.e., if $k + 1 \geq d$. This is an algebraic analog of the Spanier–Whitehead duality (see e.g., [36] and [37] Chapter 8 Exercise F).

It is also possible to reformulate the preceding discussion in terms of the Macaulay inverse θ_J of the ideal J. Here is how this works.

Corollary 3.5. *Let $J \subset \mathbb{F}[V]$ be an $\mathfrak{m}$-primary irreducible ideal with Macaulay inverse $\theta_J \in \mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$ of degree $-d$, so $d = f\text{-}\dim(\mathbb{F}[V]/J)$. If $h(X) \in \mathbb{F}[V, X]$ is a monic polynomial in X of strictly positive degree $k+1$ choose a polynomial $\overline{h}(X) \in \mathbb{F}[V, X]$ of degree d in X such that $h(X) \cdot \overline{h}(X) = X^{k+d+1} \in \Big(\mathbb{F}[V]/J\Big)[X]$ (see the discussion preceding Lemma 3.3). Then writing $\cap$ for the action of polynomials on inverse polynomials the inverse form*

$$\theta = \overline{h}(X) \cap (\theta_J \cdot X^{-(k+d)}) = \theta_J \cdot X^{-k} + \overline{h}_1 \cap (\theta_J \cdot X^{-(k+1)}) + \cdots + \overline{h}_d \cap (\theta_J \cdot X^{-(k+d)})$$

in $\mathbb{F}[z_1^{-1}, \dots, z_n^{-1}, X^{-1}]$ defines an irreducible $\mathfrak{m}$-primary ideal $I(\theta) \subset \mathbb{F}[V, X]$ which is a projective bundle ideal with bundle dimension $k+1$ and base ideal J. The formal dimension of the corresponding Poincaré duality quotient $\mathbb{F}[V, X]/I(\theta)$ is $d + k$.

Proof. This follows from Theorems 3.4 and 2.5. □

4. Connected sums of Poincaré duality algebras

Recall that for two Poincaré duality algebras H' and H'' of the same formal dimension their **connected sum**, denoted by $H'\#H''$, is defined in the following way: The homogeneous components of the graded vector space $H'\#H''$ are

$$(H'\#H'')_k = \begin{cases} \mathbb{F} \cdot [H'\#H''] & \text{if } k = d \\ H'_k \oplus H''_k & \text{if } 0 < k < d \\ 1 \cdot \mathbb{F} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The products of two elements in either H' or H'' are as before, modulo the identification of the three fundamental classes $[H']$, $[H'\#H'']$, $[H'']$ in the socle degree. The product of an element of H' of positive degree and of H'' of positive degree is zero. The operation # turns the isomorphism classes of Poincaré duality algebras of a fixed formal dimension d over a fixed ground field $\mathbb{F}$ into a commutative torsion free monoid (see e.g., [33,34]).

One says that a Poincaré duality algebra H is **#-decomposable** if there are two nontrivial Poincaré duality algebras H' and H'' such that $H \cong H'\#H''$, otherwise one says that H is **#-indecomposable**. The #-indecomposables are generators for the Grothendieck group of the monoid of isomorphism classes of Poincaré duality algebras of a given formal dimension under the connected sum operation. The Grothendieck group of standard graded Poincaré duality algebras of formal dimension d is of interest as a means of classification since every standard graded Poincaré duality algebra can be written in an essentially unique way as a connected sum of #-indecomposable Poincaré duality algebras (see [34] Proposition 3.1).

Let us make use of these ideas together with Theorem 2.5, Corollary 3.5, and the results of [34] to describe all the threefolds arising from projective bundle ideals with $\mathbb{F}_2$ as ground field. Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal with base ideal $J = I \cap \mathbb{F}[V, X]$. Then, by Theorem 2.5 and Corollary 3.5, I is irreducible if and only if J is. If the bundle dimension is two then $\mathbb{F}[V]/J$ is a surface algebra. So, if $\mathbb{F} = \mathbb{F}_2$, then Theorem 2.5 and Proposition 3.1 of [34] says there are unique integers t and r such that

$$\mathbb{F}_2[V]/J \cong \underbrace{\mathbb{T}\#\cdots\#\mathbb{T}}_{\longleftarrow t \longrightarrow}\#\underbrace{\mathbb{P}\#\cdots\#\mathbb{P}}_{\longleftarrow r \longrightarrow},$$

where $\mathbb{T} \cong \mathbb{F}_2[x, y]/(x^2, y^2)$ is the $\mathbb{F}_2$-cohomology of a torus $S^1 \times S^1$ and $\mathbb{P} \cong \mathbb{F}_2[u]/(u^3)$ the $\mathbb{F}_2$-cohomology of a projective plane. Note that $\mathbb{F}_2[V]/J$ is then the $\mathbb{F}_2$-cohomology of the closed surface

$$M = \underbrace{(S^1 \times S^1)\#\cdots\#(S^1 \times S^1)}_{\longleftarrow t \longrightarrow} \# \underbrace{\mathbb{RP}(2)\#\cdots\#\mathbb{RP}(2)}_{\longleftarrow r \longrightarrow}.$$

The 2-plane bundles over M have only a first and second Stiefel–Whitney class, which algebraically means the homogenizing form for I is monic, homogeneous, and quadratic in X viz., $h(X) = X^2 + w_1 X + w_2 \in \mathbb{F}_2[V][X]$, where $w_1 \in \mathbb{F}_2[V]_1$ and $w_2 \in \mathbb{F}_2[V]_2$. Thus we have proven the algebraic part of the following result: The topological assertion will be dealt with in the Appendix.

Proposition 4.1. *Let $I \subset \mathbb{F}_2[V, X]$ be a $\mathfrak{m}$-primary irreducible projective bundle ideal with bundle dimension two and $f\text{-}\dim(\mathbb{F}[V, X]/I) = 3$ with base ideal $J = I \cap \mathbb{F}_2[V]$. Then there exist unique integers t and r, not both zero, and a basis $x_1, \dots, x_t, y_1, \dots, y_t, u_1, \dots, u_r$ for V together with, a linear form $w_1 \in \mathbb{F}_2[V]/J$, and a quadratic form $w_2 \in \mathbb{F}_2[V]/J$, such that I is generated by the forms*

$$\begin{array}{c} x_1^2, \dots, x_n^2, y_1^2, \dots, y_n^2, \\ x_i \cdot y_j \quad \textit{for } 1 \le i \ne j \le t, \\ x_i \cdot u_j \quad \textit{for } 1 \le i \le t \textit{ and } 1 \le j \le r, \\ y_i \cdot u_j \quad \textit{for } 1 \le i \le t \textit{ and } 1 \le j \le r, \\ X^2 + w_1 \cdot X + w_2. \end{array}$$

The base ideal J is generated by all the previous forms except for $X^2 + w_1 \cdot X + w_2$ which is the homogenizing form for I over J. The corresponding Poincaré duality quotient algebra $\mathbb{F}[V, X]\big/I$ is isomorphic to

$$\Big(\underbrace{(\mathbb{T}\#\cdots\#\mathbb{T})}_{\longleftarrow t \longrightarrow} \# \underbrace{(\mathbb{P}\#\cdots\#\mathbb{P})}_{\longleftarrow r \longrightarrow}\Big)[X]\big/\big(X^2 + w_1 \cdot X + w_2\big),$$

which is the $\mathbb{F}_2$-cohomology of the projective space bundle of a 2-plane bundle ξ over the closed surface

$$\underbrace{(S^1 \times S^1)\#\cdots\#(S^1 \times S^1)}_{\longleftarrow t \longrightarrow} \# \underbrace{\mathbb{RP}(2)\#\cdots\#\mathbb{RP}(2)}_{\longleftarrow r \longrightarrow}$$

whose Stiefel–Whitney classes are w_1 and w_2. $\square$

The base ideal of an $\mathfrak{m}$-primary irreducible projective bundle ideal I in $\mathbb{F}_2[V, X]$ of bundle dimension two and quotient of formal dimension three contains no nonzero linear forms, so the homogeneous component of degree one of the quotient algebra has dimension $2t + r$, where t and r are as in Proposition 4.1. Hence the Poincaré polynomial of the quotient is of the form $1 + (2t + r) \cdot Z + (2t + r) \cdot Z^2 + Z^3$. Any integer $s \in \mathbb{N}$ can be written in at least one way as $2t + r$ for $t, r \in \mathbb{N}$, so it follows that any monic, palindromic, cubic occurs as the Poincaré polynomial of an $\mathfrak{m}$-primary irreducible projective bundle ideal I in $\mathbb{F}_2[V, X]$ of bundle dimension two with quotient of formal dimension three. The algebras in Proposition 4.1, apart from the single case $\mathbb{F}_2[x, y]\big/(xy, x^3 + y^3) \cong H^*(\mathbb{RP}(3)\#\mathbb{RP}(3); \mathbb{F}_2)$, (see Example 1 in Section 1) are all #-indecomposable as will be shown in Corollary 4.3 that follows. The only projective bundle ideals in $\mathbb{F}_2[V, X]$ with quotient of formal dimension three not covered by Proposition 4.1 are isomorphic to $H^*(S^1 \times S^1 \times S^1); \mathbb{F}_2)$, $H^*(\mathbb{S}^1 \times \mathbb{RP}(2); \mathbb{F}_2)$ (the case of a 3-plane bundle over S^1) or $H^*(\mathbb{RP}(3); \mathbb{F}_2)$ (the case of a 4-plane bundle over a point).

It seems very difficult to write down a minimal generating set for the Grothendieck group of standard graded Poincaré duality algebras of formal dimension $d > 2$. There are simply too many #-indecomposables to account for, as is implied by the following discussion, for which we need to introduce another concept borrowed from topology.

Let A be a commutative graded algebra over a field and $S \subset A$ a graded subset. The **$\times$-length**[18] of S is the smallest integer c_S such that the product of any $c_S + 1$ elements of S is zero in A if such an integer c_S exists, otherwise we say the $\times$-length of S is infinite.

Proposition 4.2. *Let H be a standard graded Poincaré duality algebra of formal dimension d. Suppose there is a codimension one subspace $V \subset H_1$ of $\times$-length strictly less than d. Then, either*

(i) *H is indecomposable with respect to the connected sum operation #, or*
(ii) *H has rank two and $H \cong \mathbb{F}[x, y]\big/(xy, x^d - y^d) = \big(\mathbb{F}[x](x^{d+1})\big)\#\big(\mathbb{F}[y]\big/(y^{d+1})\big)$.*

Proof. Suppose that $H = H'\#H''$ is a nontrivial connected sum. Let the rank of H be r, that of H' be r', and that of H'' be r'', so $r = r' + r''$. Recall the formula from linear algebra relating the dimensions of two subspaces U', U'' of an $\mathbb{F}$-vector space U, viz.,

$$\dim_{\mathbb{F}}(U' + U'') = \dim_{\mathbb{F}}(U') + \dim_{\mathbb{F}}(U'') - \dim_{\mathbb{F}}(U' \cap U'').$$

Apply this to $V, H_1' \subset H_1$. After a slight rearrangement one obtains

$$\dim_{\mathbb{F}}(V + H_1') + \dim_{\mathbb{F}}(V \cap H_1') = \dim_{\mathbb{F}}(V) + \dim_{\mathbb{F}}(H_1') = r - 1 + r'.$$

On the other hand we have the inequality

$$\dim_{\mathbb{F}}(V + H_1') + \dim_{\mathbb{F}}(V \cap H_1') \leq r + \dim_{\mathbb{F}}(V \cap H_1'),$$

so

$$r + \dim_{\mathbb{F}}(V \cap H_1') \geq r + r' - 1,$$

whence we conclude that

$$\dim_{\mathbb{F}}(V \cap H_1') \geq r' - 1.$$

Since $V \cap H_1' \subset V$ the subalgebra of H generated by $V \cap H_1'$ has $\times$-length at most $d - 1$. If $\dim_{\mathbb{F}}(V \cap H_1')$ were to equal r' then, since H' is a Poincaré duality algebra of formal dimension d, this would imply that H_1' was trivial since no product of d elements of H_1' could be nonzero. Hence $\dim_{\mathbb{F}}(V \cap H_1') = r' - 1$. This tells us that $V \cap H_1'$ is a codimension subspace of H_1'. By symmetry $V \cap H_1'' \subset H_1''$ is also a codimension one subspace of $\times$-length at most $d - 1$.

Putting these facts together says that $(V \cap H_1') \oplus (V \cap H_1'') \subset V$ is a codimension one subspace. So we may choose a $v \in V$ that does not belong to this subspace. Write $v = v' + v''$, with $v' \in H_1'$ and $v'' \in H_1''$. (Recall that $H_1' \oplus H_1'' = V$.) Note that $v' \notin V \cap H_1'$: For if $v' \in V \cap H_1'$, then $v'' = v - v' \in V \cap H_1''$ which implies that $v = v' + v''$ belongs to $(V \cap H_1') \oplus (V \cap H_1'')$ contrary to how we chose v. Therefore $v' \notin V \cap H_1'$ and similarly $v'' \notin V \cap H_1''$.

[18] An algebraic topologist would probably call this the $\cup$-length (pronounced *cup length*). In topology $\cup$-length provides a lower bound for the *category* of a topological space, i.e., the number of contractible subsets needed to cover a space.

Retaining these notations we next choose a basis $v_1, \ldots, v_{r'-1}$ for $V \cap H_1'$. Note that v' extends this to a basis for H_1'. Consider a product $v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k}$ of d elements from this basis. One has

$$v_{i_1} \cdots v_{i_k} \cdot (v')^{d-k} = v_{i_1} \cdots v_{i_k} \cdot (v' + v'')^{d-k} = v_{i_1} \cdots v_{i_k} \cdot v^{d-k} = 0$$

for $k > 0$, since $v'' \in H_1''$ annihilates $v_1, \ldots, v_{r'-1} \in H_1'$, and $v_{i_1} \cdots v_{i_k} \cdot v^{d-k}$ is a product of d elements of V which has $\times$-length at most $d-1$. Thus the only product of d elements of the basis $v_{i_1}, \ldots, v_{i_k}, v'$ for H_1' that is nonzero is $(v')^d$. Poincaré duality then forces that H' has rank one and is isomorphic to $\mathbb{F}[x]\big/(x^{d+1})$. Likewise $H'' \cong \mathbb{F}[y]\big/(y^{d+1})$. Finally one notes that $H = \left(\mathbb{F}[x]\big/(x^{d+1})\right)\#\left(\mathbb{F}[y]\big/(y^{d+1})\right)$ satisfies the hypotheses of the proposition: Namely the subspace spanned by $x+y$ in H_1 has codimension one and $\times$-length $d-1$. $\square$

Corollary 4.3. *Let $I \subset \mathbb{F}[V, X]$ be a projective bundle ideal with a Poincaré duality quotient algebra of formal dimension m. Then either*

(i) *$\mathbb{F}[V, X]\big/I$ is #-indecomposable, or*
(ii) *$\mathbb{F}[V, X] \cong \mathbb{F}[x, y]\big/(xy, x^{m+1} - y^{m+1})$.*

Proof. Let $J = \mathbb{F}[V] \cap I$ be the base ideal and $\mathbb{F}[V]\big/J$ have formal dimension d. Consider the coexact sequence

$$\mathbb{F} \longleftarrow \mathbb{F}[X]\big/(X^{k+1}) \longleftarrow \mathbb{F}[V, X] \longleftarrow \mathbb{F}[V]\big/J \longleftarrow \mathbb{F}.$$

By Lemma 2.1 the formal dimension of $\mathbb{F}[V, X]\big/I$ is $d+k$, so the codimension one subspace V of $\left(\mathbb{F}[V, X]\big/I\right)_1$ has $\times$-length at most $d < m = d + k$ and the result follows from Proposition 4.2. $\square$

We close this section with a result generously offered by the referee, showing that the *generic* Poincaré duality algebra of formal dimension $d \geq 3$ and rank $r \geq 4$ over an infinite field is #-indecomposable.

Proposition 4.4. *Let $\mathbb{F}$ be an infinite field. Then the isomorphism classes of Poincaré duality algebras of formal dimension d and rank r has dimension $\binom{r+d-1}{d}$ as an algebraic set, whereas the dimension of those which are the connected sum of an algebra of rank a and an algebra of rank $b = r - a$ is $\binom{a+d-1}{d} - a^2 + \binom{b+d-1}{d} - b^2$. Since for $d \geq 3$ and $r \geq 4$ the former is strictly larger than the latter the generic Poincaré duality algebra of formal dimension d and rank r in such a case is #-indecomposable.*

Proof. By Macaulay's Theorem (see e.g., [16] Theorem II.4.2 and Theorem VI.1.1) the isomorphism classes of Poincaré duality algebras of formal dimension d and rank r is in bijective correspondence with the quotient of the projective space of the vector space of inverse forms $\mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]_{-d}$ of degree $-d$ by the action of the projective linear group $\mathrm{PGL}(n, \mathbb{F})$. The computation of the dimensions then follows. $\square$

5. The Poincaré duality algebra dual to an element

The motivation for the constructions in this section come to us from topology. If $\lambda \downarrow M$ is a line bundle over a closed smooth manifold with first Stiefel–Whitney class w_1 then a **manifold dual to** λ or **to** w_1 is obtained as follows: Choose a classifying map $f_\lambda : M \longrightarrow \mathbb{RP}(\ell)$ for some large integer ℓ which is transverse regular to $\mathbb{RP}(\ell - 1)$. The manifold N which is the preimage $f_\lambda^{-1}(\mathbb{RP}(\ell-1))$ is said to be dual to λ or to w_1. It can be thought of as the set of zeros of a generic section to λ. The normal bundle of N in M is λ. This is the process of **dualizing a line bundle** referred to in the introduction.

Algebraically, this corresponds to the following simple construction. Let H be a Poincaré duality algebra and $0 \neq u \in H$. Then the trivial ideal $(0) \subset H$ is irreducible, so the Noether Involution Theorem tells us that $\mathrm{Ann}_H(\mathrm{Ann}_H(u)) = u$ for any nonzero element $u \in H$ and that an ideal $J \subset H$ is irreducible if and only if $\mathrm{Ann}_H(J)$ is a principal ideal. Hence $\mathrm{Ann}(u) \subset H$ is an irreducible ideal. The algebra $H\big/\mathrm{Ann}_H(u)$ is a Poincaré duality algebra of formal dimension f-$\dim(H) - \deg(u)$ (see e.g., [16] Corollaries I.2.2–I.2.4) which we call the **dual of** u **in** H. This type of construction is well known to commutative algebraists (see e.g., [44]), as was pointed out to us by the referee. Dualizing a line bundle corresponds to the special case where u has degree one.

Despite being simple, the dualizing construction can have surprising consequences. Again, we illustrate this with some unusual examples. We choose $\mathbb{F}_2$ as the ground field in these examples to simplify the arithmetic.

Example 1. Consider the inverse form $\theta_s = (x^{-1} + y^{-1})^s \in \mathbb{F}_2[x^{-1}, y^{-1}]$ for $s \in \mathbb{N}$. To compute the corresponding ideal $I(\theta_s)$ we introduce the auxiliary form

$$\theta_s y^{-t} = x^{-s} y^{-t} + \cdots + y^{-(s+k)}$$

which is of the type considered in Example 1 in Section 2. Write $s = 2^a + b$ with $0 \leq b < 2^a$, and let $c = 2^a - b$. As in that example, the ideal $I(\theta_s y^{-t})$ is generated by the two forms x^{s+1} and $y^{t+1-k}(y^k + \alpha_1 x y^{k-1} + \cdots \alpha_k x^k)$, where $\alpha_1, \ldots, \alpha_k$ are determined by the equation

$$\begin{aligned} 1 + \alpha_1 x + \cdots + \alpha_k x^k &= \frac{1}{(1+x)^s} = \frac{(1+x)^{2^{s+1}}}{(1+x)^{2^a+b}} = (1+x)^{2^a-b} = (1+x)^c \\ &= 1 + \binom{c}{1} x + \cdots + \binom{c}{c} x^c \end{aligned}$$

after equating coefficients of powers of x. In the case that $t = c - 1$, this tells us that the ideal $I(\theta_s y^{c-1})$ is generated by the two forms x^{s+1} and $(x+y)^c$. Next, note that

$$x^{s+1} = x^{2^a+b+1}x^{b+1}\left((x+y)^{2^a} + (x+y)^{2^a}\right)$$

so the ideal $I(\theta_s y^{c-1})$ is also generated by the two forms $(x+y)^c$ and $x^{b+1}y^{2^a}$. If we dualize y^{c-1} in the quotient algebra $\mathbb{F}_2[x, y]\big/I(\theta_s y^{c-1})$ we obtain $I(\theta_s)$, and [32] Proposition 2.2 tells us this is the ideal generated by $(x+y)^c$ and $(xy)^{b+1}$

If $s = 2^t - 1$, then $c = 1$, so the resulting ideal $I(\theta_s)$ contains the linear form $x + y$. If in addition s is divisible by r, say, and $(2^t - 1)/r$ is not itself a power of two minus one,then the ideal $I(\theta_{s/r})$ contains no nonzero linear forms, as the value of c for $(2^t - 1)/r$ is not one. Since θ_s is the r-th power of $\theta_{s/r}$ we obtain many examples of inverse forms θ for which the ideal $I(\theta^r)$ is not contained in $I(\theta)$. This should be contrasted with the results of [31] that imply $I(\theta^{2^i}) \subset I(\theta)$ for any nonzero inverse form θ in characteristic two and any $i \in \mathbb{N}_0$. For more about the interaction of the Frobenius map with irreducible ideals and their Macaulay inverses see [16] Section II.6.

Example 2. Consider the ideal $I = (e_1, \ldots, e_n)$ in $\mathbb{F}_2[z_1, \ldots, z_n]$ generated by the elementary symmetric polynomials $e_1, \ldots, e_n$. The Macaulay inverse for this ideal is

$$\theta = \sum_{\sigma \in \Sigma_n} z_{\sigma(1)}^{0} z_{\sigma(2)}^{-1} \cdots z_{\sigma(n)}^{-(n-1)},$$

see [11] Section 4 Example 2. By Sharp's Theorem (cf [16] Theorem II.6.5) the Frobenius square $I^{[2]}$ is $\mathfrak{m}$-primary, irreducible, and by [16] Theorem II.6.6 has as Macaulay inverse of the form $z_1^{-1} \cdots z_n^{-1}\theta^2$. If we dualize $z_1 \cdots z_n \in H = \mathbb{F}_2[z_1, \ldots, z_n]\big/I^{[2]}$ then Proposition 6.1 tells us that

$$(z_1 \cdots z_n) \cap (z_1^{-1} \cdots z_n^{-1}\theta^2) = \theta^2 = \sum_{\sigma \in \Sigma_n} z_{\sigma(1)}^{0} z_{\sigma(2)}^{-2} \cdots z_{\sigma(n)}^{-2(n-1)}$$

is the Macaulay dual for the ideal of $\mathbb{F}_2[z_1, \ldots, z_n]$ which is the kernel of the natural epimorphism of $\mathbb{F}_2[z_1, \ldots, z_n]$ onto $H\big/\mathrm{Ann}_H(z_1 \cdots z_n)$. The ideal $I(\theta^2)$ contains $I^{[2]} = (e_1^2, \ldots, e_n^2)$ and, by [31] Corollary 4.2, $I(\theta^2) = (I^{[2]} : z_1 \cdots z_n)$. Since $z_1 \cdots z_n$ divides e_n it follows from [31] Lemma 2.1 that $(I^{[2]} : z_1 \cdots z_n) = (e_1^2, \ldots, e_{n-1}^2, e_n)$. Therefore the dual of $z_1 \cdots z_n = e_n$ in $\mathbb{F}_2[z_1, \ldots, z_n]\big/I^{[2]}$ turns out to be $I(\theta^2) = (e_1^2, \ldots, e_{n-1}^2, e_n)$, and more generally $I(\theta^{2^r}) = (e_1^{2^r}, \ldots, e_{n-1}^{2^r}, e_n^{2^r-1})$. The element $z_1 \cdots z_n$ is a Thom class so, by [16] Theorem III.1.4, all these ideals are invariant under the Steenrod algebra $\mathscr{A}^*$ of $\mathbb{F}_2$.

Example 3. There is a variation of the preceding example that arises because the algebra $\mathbb{F}_2[z_1, \ldots, z_n]\big/(e_1, \ldots, e_n)$ is not really a rank n algebra: It has rank $n - 1$ since $e_1 = z_1 + \cdots + z_n$. The images of $e_2, \ldots, e_n$ in the quotient algebra $\mathbb{F}_2[z_1, \ldots, z_{n-1}] = \mathbb{F}_2[z_1, \ldots, z_n]/(e_1)$ are denoted by $w_2, \ldots, w_n$. One has

$$w_i = \begin{cases} \bar{e}_i + \bar{e}_1\bar{e}_{i-1} & \text{for } i = 2, \ldots, n-1 \text{ and} \\ \bar{e}_1\bar{e}_{n-1} & \text{for } i = n, \end{cases}$$

where $\bar{e}_1, \ldots, \bar{e}_n \in \mathbb{F}_2[z_1, \ldots, z_{n-1}]$ are the elementary symmetric polynomials in the variables $z_1, \ldots, z_{n-1}$. A Macaulay inverse for the ideal generated by $w_2, \ldots, w_n$ is

$$\psi = \sum_{\sigma \in \Sigma_{n-1}} z_{\sigma(1)}^{-1} \cdots z_{\sigma(n-1)}^{-(n-1)}$$

(see [16] Section VI.4). So if we dualize $z_1 \cdots z_{n-1}$ in $\mathbb{F}_2[z_1, \ldots, z_{n-1}]\big/(w_2, \ldots, w_n)$ we find by the same reasoning as in Example 2 that it is the quotient algebra of $\mathbb{F}_2[z_1, \ldots, z_{n-1}]$ by the ideal $I(\psi^2) = (\bar{e}_2^2 + \bar{e}_1^4, \ldots, \bar{e}_{n-1}^2\bar{e}_1^2\bar{e}_{n-2}^2, \bar{e}_1^2\bar{e}_{n-1})$. Note that the generator of maximal degree, viz. $\bar{e}_1^2\bar{e}_{n-1}$ is not a polynomial in the generators of the original ideal $(w_2, \ldots, w_n) = I(\psi)$, a phenomenon not seen in examples before. Again $I(\psi^{2^r})$ is a family of $\mathscr{A}^*$-invariant ideals.

6. Pulling off sections and inverse symmetric forms

This section is devoted to what we find is an interesting family of standard graded Poincaré duality algebras which are not generic, arising from a construction related to the topological notion of pulling sections off a vector bundle. Note that in the equation

$$\theta_I = \bar{h} \cap (\theta_J \cdot X^{-(k+d)})$$

occurring in Corollary 3.5 one has

$$\deg(\bar{h}(X)) = d, \qquad \deg(\theta_J \cdot X^{-(k+d)}) = -2d - k.$$

Since $\deg(\bar{h}(X)) = d = -\deg(\theta_J)$ the formula

$$\theta_m = \bar{h}(X) \cap (\theta_J \cdot X^{-m})$$

defines an inverse form of degree $-m$ for any integer $m \in \mathbb{N}$. One may consider these inverse forms (if nonzero) as defining additional $\mathfrak{m}$-primary irreducible ideals in $\mathbb{F}[V, X]$ which may not be projective bundle ideals at all.

Topologically, for $m \leq k + 1$, this would correspond to pulling sections off of a vector bundle. We use this terminology in the algebraic context also. For example, the bundle $\xi \downarrow \mathbb{RP}(n-1)$ considered in connection with Example 1 of Section 1 could have been taken to be a line bundle, rather than a 2-plane bundle, since in the algebraic context of this example the only feature of ξ we used was the total Stiefel–Whitney class, which was $1 + z \in H^*(\mathbb{RP}(n-1); \mathbb{F}_2) = \mathbb{F}_2[z]\big/(z^n)$.

Pulling off sections in the sense just described amounts to dualizing powers of X in the algebra $\mathbb{F}[V, X]\big/(J, h(X))$, as described in Section 5. Here is why.

Proposition 6.1. *Let $I(\theta) \subset \mathbb{F}[z_1, \ldots, z_n]$ be the $\mathfrak{m}$-primary irreducible ideal defined by the inverse form $\theta \in \mathbb{F}[z_1, \ldots, z_n]$. If $u \neq 0 \in \mathbb{F}[z_1, \ldots, z_n]\big/I(\theta)$ then the inverse form $u \cap \theta$ is a Macaulay dual for the ideal K in $\mathbb{F}[z_1, \ldots, z_n]$ defining the dual to u in $\mathbb{F}[z_1, \ldots, z_n]\big/I(\theta)$.*

Proof. Let $J = I(u \cap \theta) \subset \mathbb{F}[z_1, \ldots, z_n]$. Then J is $\mathfrak{m}$-primary and irreducible. The corresponding Poincaré duality quotient algebra has formal dimension $|\deg(\theta)| - \deg(u)$. If $f \in J$ then $\theta(fu) = (u \cap \theta)(f) = 0$, so $fu \in I(\theta)$. This means there is an induced epimorphism

$$\varphi : \mathbb{F}[z_1, \ldots, z_n]\big/J \longrightarrow \Big(\mathbb{F}[z_1, \ldots, z_n]\big/I(\theta)\Big)\Big/\mathrm{Ann}_{\mathbb{F}[z_1,\ldots,z_n]/I(\theta)}(u) = \mathbb{F}[z_1, \ldots, z_n]\big/K.$$

The algebra $\mathbb{F}[z_1, \ldots, z_n]\big/I(\theta)$ also has formal dimension $|\deg(\theta)| - \deg(u)$ by [16] Corollary I.2.3, and hence φ must be an isomorphism, [16] Corollary I.2.4, making the inclusion $I(u \cap \theta) \subseteq K$ an equality. □

In a topological context the geometric dimension of a bundle[19] imposes a restriction on how many sections one may pull off (see e.g. [15], where this topological restriction was of central importance). Algebraically, however, there is only the restriction imposed by the requirement that $\theta_m = \bar{h}(X) \cap (\theta_J \cdot X^{-m})$ be an actual nonzero inverse form, so $m \in \mathbb{N}$. We exploit this next.

We start with an inverse form $\theta_J \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ of degree $-d$ and a (dual homogenizing) form $\varphi(X) = X^d + \varphi_1 X^{d-1} + \cdots + \varphi_d \in \mathbb{F}[V, X]$ of degree d and consider the homogenizations of θ_J given by

$$\theta = \varphi(X) \cap (\theta_J \cdot X^{-m}) = \theta_J \cdot X^{-m} + \varphi_1 \cap (\theta_J \cdot X^{-(m+1)}) + \cdots + \varphi_d \cap (\theta_J \cdot X^{-(m+d)}) \in \mathbb{F}[V^{-1}, X^{-1}],$$

for $m \in \mathbb{N}$. These define $\mathfrak{m}$-primary irreducible ideals $I(\theta_m) \subset \mathbb{F}[V, X]$. We organize this section around an extensive family of ideals arising in this way, and use them to bring to the fore a number of the less obvious[20] properties of $\mathfrak{m}$-primary irreducible ideals.

Notation. We denote by Σ_n the symmetric group which acts on $\mathbb{F}[z_1, \ldots, z_n]$ by permutation of the variables. Let $e_1, \ldots, e_n \in \mathbb{F}[z_1, \ldots, z_n]$ be the elementary symmetric polynomials in the variables $z_1, \ldots, z_n$, and $\sigma_1, \ldots, \sigma_n \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}]$ their analogs as inverse polynomials. Specifically, σ_i is the Σ_n-orbit sum $\mathfrak{S}(z_1^{-1} z_2^{-2} \cdots z_i^{-i})$ (i.e., the sum of all the elements in the orbit) of the inverse monomial $z_1^{-1} z_2^{-2} \cdots z_i^{-i}$. By convention $e_0 = 1 = \sigma_0$.

With these notations one has

$$\sigma_i = e_{n-i} \cap \sigma_n \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}] \quad \text{for } i = 0, \ldots, n.$$

Introduce the form $\varphi(X) = e_n + e_{n-1}X + \cdots + e_1 X^{n-1} + X^n$ and define the inverse polynomial θ_n by

$$\begin{aligned} \theta_n &= \varphi(X) \cap (\sigma_n \cdot X^{-n}) = (e_n + e_{n-1}X + \cdots + e_1 X^{n-1} + X^n) \cap (\sigma_n \cdot X^{-n}) \\ &= X^{-n} + \sigma_1 \cdot X^{-(n-1)} + \cdots + \sigma_{n-1} X^{-1} + \sigma_n = \prod_{i=1}^{n} (X^{-1} + z_i^{-1}) \in \mathbb{F}[z_1^{-1}, \ldots, z_n^{-1}, X^{-1}]. \end{aligned}$$

Note that θ_n has degree $-n$ so the corresponding Poincaré duality quotient algebra has formal dimension n.

Notation. The Poincaré duality quotient $\mathbb{F}[z_1, \ldots, z_n, X]\big/I(\theta_n)$ will be denoted by $P(n)$.

If $I(\sigma_n) \subset \mathbb{F}[z_1, \ldots, z_n]$ denotes the $\mathfrak{m}$-primary irreducible ideal that is the Macaulay dual to σ_n then all squares of elements in $Q(n) = \mathbb{F}[z_1, \ldots, z_n]\big/I(\sigma_n)$ are zero since no monomial divisible by the square of one of the variables

19 The **geometric dimension** of a vector bundle ξ over a topological space B is the minimal integer m such that there is a vector bundle $\zeta \downarrow B$ stable equivalent to ξ.

20 It is very likely that many of these properties were known to F.S. Macaulay, but due to the enormous change in terminology that has taken place since he wrote [13] this is very difficult to confirm this by reference to it.

$z_1, \ldots, z_n$ occurs in the support[21] of σ_n. The algebra $Q(n)$ is therefore a quotient of the **exterior algebra** $E(z_1, \ldots, z_n)$ generated by $z_1, \ldots, z_n$. Since both these algebras are Poincaré duality algebras of formal dimension n the quotient map $E(z_1, \ldots, z_n) \longrightarrow Q(n)$ must be an isomorphism (see e.g., [16] Lemma I.3.1). Hence $Q(n) = E(z_1, \ldots, z_n)$ is an exterior algebra on the generators $z_1, \ldots, z_n$. Observe that $f \cap \theta_n = 0$ for any $f \in I(\sigma_n)$, so there is a natural map

$$\mathbb{F}[z_1, \ldots, z_n]\big/I(\sigma_n) \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]\big/I(\theta_n)$$

which may be extended to a map

$$\big(\mathbb{F}[z_1, \ldots, z_n]\big/I(\sigma_n)\big)[X] \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]\big/I(\theta_n)$$

in the obvious way. Hence we have shown the following.

Lemma 6.2. *With the preceding notations one has $z_i^2 = 0 \in P(n)$ for $i = 1, \ldots, n$.* □

Before we delve deeper into the structure of the ideals $I(\theta_n)$ we indicate how these ideals arise from projective bundle ideals by stripping off sections as described above. To simplify this discussion we *assume that the ground field has characteristic* 2 so

$$\varphi(X)^2 = X^{2n} \in \big(\mathbb{F}[z_1, \ldots, z_n]\big/I(\sigma_n)\big)[X]$$

by Lemma 6.2. Therefore, from the $\mathfrak{m}$-primary ideal $I(\sigma_n) \subset \mathbb{F}[z_1, \ldots, z_n]$ and the homogenizing polynomial $\varphi(X)$ we obtain a projective bundle ideal $(I(\sigma_n), \varphi(X)) \subset \mathbb{F}[z_1, \ldots, z_n, X]$ with base ideal $I(\sigma_n)$ and bundle dimension n. A Macaulay dual for this ideal is

$$\varphi(X) \cap (\sigma_n \cdot X^{2n-1})$$

since $\varphi(X)$ serves as its own dual homogenizing form.[22] The Macaulay inverse θ_n of the ideal $I(\theta_n)$ may be thought of as arising from the projective bundle ideal $(I(\sigma_n), \varphi(X))$ by stripping off n-sections.

Lemma 6.3. *The Poincaré duality algebra $P(n)$ has rank $n+1$, i.e., $P(n)_1$ has dimension $n+1$ as an $\mathbb{F}$-vector space.*

Proof. The value of θ_n on e_n is 1, which shows for $i = 1, \ldots, n$ that $z_1 \cdots \hat{z}_i \cdots z_n$ serves as a Poincaré dual for $z_i \in P(n)$ and is zero on z_j for $j \neq i$, so these elements are linearly independent. Likewise θ_n evaluates to 1 on X^n so not only is X nonzero but so are all its powers up to the n-th. Hence X cannot be a linear combination of the elements $z_1, \ldots, z_n$, because their squares are zero. □

To analyze in more detail the multiplication of $P(n)$ we employ the natural map

$$E(z_1, \ldots, z_n)[X] = \big(\mathbb{F}[z_1, \ldots, z_n]\big/I(\sigma_n)\big)[X] \longrightarrow \mathbb{F}[z_1, \ldots, z_n, X]\big/I(\theta_n) = P(n)$$

introduced above. We note that as $z_1, \ldots, z_n, X$ generate $P(n)$ as an algebra, this map is an epimorphism. So every element of $P(n)$ may be written as a sum of monomials of the form $X^t z_S$, where $S = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$, $z_S = z_{i_1} \cdots z_{i_s}$, and $s = |S|$ the number of elements in S. Since $P(n)$ has formal dimension n, we need only consider such monomials for $t + |S| \leq n$. For two such monomials $X^{t_1} z_{S_1}$ and $X^{t_2} z_{S_2}$ of complementary degree (i.e., for which $t_1 + |S_1| + t_2 + |S_2| = n$) their product is given by

$$X^{t_1} z_{S_1} \cdot X^{t_2} z_{S_2} = \begin{cases} 0 & \text{if } S_1 \cap S_2 \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Fix an integer j with $2j \leq n$. We introduce a matrix $\mathsf{M}(j, n-j)$ that encodes the products of elements of degree j with elements of degree $n-j$. The rows of the matrix are to be indexed by

$$X^j, X^{j-1}z_1, \ldots, X^{j-1}z_n, X^{j-2}z_1z_2 + X^{j-1}(z_1 + z_2), \ldots.$$

The columns are indexed by

$$X^{n-j}, X^{n-j-1}z_1 + X^{n-j}, X^{n-j-1}z_2 + X^{n-j}, \ldots, X^{n-j-|T|}z_T + X^{n-j}, \ldots, \quad |T| \leq j.$$

The general term of $\mathsf{M}(j, m-j)$ being

$$X^{j-|S|}z_S + \sum_{\varnothing \subsetneqq T \subsetneqq S} X^{j-|T|}z_T, \quad |S| \leq j,$$

so the terms get larger (in the sense they have larger support among the monomials we are using) with increasing row number. The entries of the matrix are the value of θ_n on the product of the forms indexing the rows and columns. This defines a square matrix $\mathsf{M}(j, n-j)$ of size

$$m = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{j},$$

21 By the **support** of a form we mean the set of monomials occurring with a nonzero coefficient in the form.

22 For another context in which this type of self duality was exploited see [28].

which is an analog of the catalecticant matrix (see e.g., [16] Section VI.2) The first row of $\mathsf{M}(j, n-j)$ is $(1, 0, \dots, 0)$ since $X^j X^{n-j} = 1$ and $X^j(X^{n-j-|T|}z_T + X^{n-j}) = 1 + 1 = 0$. The first column of the matrix consists entirely of 1s, since in the first row of that column one has $X^j X^{n-j} = 1$, and in the remaining rows $\left(\sum_{\varnothing \subsetneqq T \subseteq S} X^{j-|T|}z_T\right)X^{n-j}$, which is a sum of $2^{|S|} - 1$ entries each of which is 1.

Next one considers the product

$$(✣) \quad \left(\sum_{\varnothing \subsetneqq T \subseteq S} X^{j-|T|}z_T\right)\left(X^{n-j-|U|}z_U + X^{n-j}\right)$$

for subsets $S, U \subset \{1, \dots, n\}$. Note that

$$\left(X^{j-|T|}z_T\right)\left(X^{n-j-|U|}z_U + X_{n-j}\right) = \begin{cases} 1 & \text{if } T \cap U \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

so the product $(✣)$ being the sum of all these terms is the number of $\varnothing \subsetneqq T \subseteq S$ for which $T \cap U \neq \varnothing$. If $U \supseteq T$ this sum is $2^{|S|} - 1 \neq 0 \in \mathbb{F}$ and if $U \cap S = \varnothing$ the sum is 0. So it remains to consider the situation $\varnothing \neq U \cap S \neq S$, where one finds that the nonempty sets T with $T \cap U = \varnothing$ are those contained in $S \setminus (U \cap S)$. So their number is

$$2^{|S|} - 1 - (2^{|U|-|U \cap S|} - 1) = 2^{|S|} - 2^{|S|-|U \cap S|}$$

which is even, and hence zero in $\mathbb{F}$ (which remember was assumed to have characteristic 2). This says that the matrix $\mathsf{M}(j, n-j)$ with rows and columns indexed as described above has the form indicated in the next table.

$\mathsf{M}(j, n-j)$	X^{n-j}	small support $\rightsquigarrow$	large support
X^j	1	$0 \cdots 0$	$0 \cdots 0$
	1	$1\ 0\ 0$	$0 \cdots 0$
small support	$\vdots$	$0 \ddots 0$	$0 \ddots 0$
	1	$0\ 0\ 1$	$0 \cdots 0$
	$\vdots$	$0 \cdots 0$	$1\ 0\ 0$
large support		$0 \ddots 0$	$0 \ddots 0$
	1	$0 \cdots 0$	$0\ 0\ 1$

The matrix $\mathsf{M}(j, n-j)$ is therefore nonsingular so the monomials indexing the rows are linearly independent in $P(n)$. Hence we have proven the first assertion of the following result.

Lemma 6.4. *For $j \leq n/2$ we have*

$$\dim_{\mathbb{F}}(P(n)_j) = 1 + \binom{n}{1} + \cdots + \binom{n}{j}$$

and therefore the epimorphism

$$E(z_1, \dots, z_n)[X] \longrightarrow P(n)$$

is an isomorphism through degree $\left[\frac{n}{2}\right]$.

Proof. One only need note that $E(z_1, \dots, z_n)[X]$ also has dimension $1 + \binom{n}{1} + \cdots + \binom{n}{j}$ for $j \leq \left[\frac{n}{2}\right]$. □

From Lemma 6.4 we can deduce the interesting result that the minimum number of generators of the ideal $I(\theta_n)$ exceeds the number of variables by roughly $n!$. This shows that there are $\mathfrak{m}$-primary irreducible ideals not arising from connected sums where the number of generators is arbitrarily larger than the number of variables or the formal dimension of the corresponding Poincaré duality quotient algebra.

Proposition 6.5. *The number of linearly independent forms of degree $\frac{n}{2}$ for n even or $\frac{n-1}{2}$ for n odd needed to generate the ideal $I(\theta_n)$ is*

$$\binom{n+1}{\frac{n}{2}+1} \quad \text{if } n \text{ is even, or}$$

$$\binom{n}{\frac{n+1}{2}} \quad \text{if } n \text{ is odd}.$$

Therefore the minimum number of generators of the ideal $I(\theta_n)$ exceeds the rank of $P(n)$ by one less than the preceding numbers.

Proof. Suppose that $n = 2k$ is even. Since $P(n)$ is a Poincaré duality algebra the homogeneous components $P(n)_{k-1}$ and $P(n)_{k+1}$ must have the same dimension. By Lemma 6.4 the dimension of $P(n)_{k-1}$ is

$$1 + \binom{n}{1} + \cdots + \binom{n}{k}.$$

On the other hand the Poincaré series of the algebra $E(z_1, \ldots, z_n)[X]$ is given by

$$P(E(z_1, \ldots, z_n)[X], t) = \frac{(1+t)^n}{1-t}$$

so the homogeneous component of $E(z_1, \ldots, z_n)[X]$ of degree $k+1$ has dimension

$$1 + \binom{n}{1} + \cdots + \binom{n}{k+1}.$$

This means that the kernel of the natural map $E(z_1, \ldots, z_n)[X] \longrightarrow P(n)$ must have dimension at least

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

as claimed. The case $n = 2k + 1$ is similar and left to the reader. Finally, if one recalls that $I(\theta_n)$ requires the n quadratic generators $z_1^2, \ldots, z_n^2$, the statement about the excess of generators over the rank follows. □

Note that by Corollary 4.3 the algebras $P(n)$ are #-indecomposable in the Grothendieck group of standard graded Poincaré duality algebras. For other examples of Gorenstein ideals requiring large numbers of generators see [21].

For the remainder of this section we replace the hypothesis that $\mathbb{F}$ have characteristic two with the assumption that the ground field is finite. In this case the Steenrod algebra of the ground field acts on the polynomial algebra $\mathbb{F}[V]$ (see e.g., [24] Chapter 10 or [29]). We note that the ideals $I(\theta_n)$ are closed under the action of the Steenrod algebra and compute their conjugate Wu classes.[23] A result of S.P. Mitchell (see [18] Appendix B or [16] Part IV Section 2) implies that the conjugate Wu classes of a coinvariant algebra satisfying Poincaré duality must vanish, so the next result shows that the algebras $P(n)$ cannot be coinvariant algebras.[24]

Proposition 6.6. *Let $\mathbb{F} = \mathbb{F}_q$ be the Galois field with q elements,*

$$\theta_n = \varphi(X) \cap (\sigma_n \cdot X^{-n}) \in \mathbb{F}_q[z_1{-}1, \ldots, z_n^{-1}, X^{-1}],$$

and $I(\theta_n) \subset \mathbb{F}_q[z_1, \ldots, z_n, X]$ the $\mathfrak{m}$-primary irreducible ideal it defines. Then $I(\theta_n)$ is a $\mathcal{P}^*$-invariant ideal and the conjugate Wu classes of the unstable $\mathcal{P}^*$ quotient algebra $\mathbb{F}_q[z_1, \ldots, z_n, X]\big/I(\theta_n)$ are given by the formula

$$\chi\mathrm{Wu} = (1 + X^{q-1})^n \cdot \prod_{i=1}^{n}(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1}).$$

Proof. By [16] Theorem VI.6.2 one has $\boldsymbol{\mathcal{P}}(\theta_n) = \chi\mathrm{Wu}(P(n)) \cdot \theta_n$, where $\boldsymbol{\mathcal{P}} = 1 + \mathcal{P}^1 + \cdots + \mathcal{P}^k + \cdots$ is the formal sum of the reduced power operations. To compute $\boldsymbol{\mathcal{P}}(\theta_n)$ we note that by the mixed Cartan formula (see the discussion preceding Theorem VI.6.2 in [16])

$$(\maltese) \quad \boldsymbol{\mathcal{P}}(\theta_n) = \boldsymbol{\mathcal{P}}(\varphi_n(X)) \cap \big(\boldsymbol{\mathcal{P}}(\sigma_n) \cdot \boldsymbol{\mathcal{P}}(X^{-n})\big)$$

so we turn to a computation of the individual factors.

We have

$$\varphi_n(X) = e_n + e_{n-1}X + \cdots e_1X^{n-1} + X^n = \prod_{i=1}^{n}(X + z_i)$$

is a product of linear forms and hence a Thom class. Therefore one obtains

$$\begin{aligned}(\bigstar) \quad \boldsymbol{\mathcal{P}}(\varphi_n(X)) &= \prod_{i=1}^{n}\boldsymbol{\mathcal{P}}(X + z_i) = \prod_{i=1}^{n}(X + z_i + X^q + z_i^q) \\ &= \prod_{i=1}^{n}\big((X + z_i) + (X + z_i)(X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1})\big) \\ &= \prod_{i=1}^{n}(X + z_i)\prod_{i=1}^{n}(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1}) \\ &= \varphi_n(X)\prod_{i=1}^{n}(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1}).\end{aligned}$$

The action of the total reduced power operation $\boldsymbol{\mathcal{P}}$ on the inverse monomial X^{-n} is given by (see e.g., [16] Proposition 6.5.2 and the discussion following its proof)

$$(\ast) \quad \boldsymbol{\mathcal{P}}(X^{-n}) = X^{-n}(1 + X^{-1})^n.$$

[23] For a discussion of topological Wu classes see e.g., [1] or [42] pp 98–99 and 122–123 and for the algebraic case over finite fields see [10], [16] Section III.3.

[24] It is an open question if a coinvariant algebra satisfying Poincarë duality must be a complete intersection. This is one reason we have computed the conjugate Wu classes of $P(n)$.

The action of $\mathscr{P}$ on the inverse monomial σ_n is trivial, i.e.,

$$(\circledast) \quad \mathscr{P}(\sigma_n) = \sigma_n.$$

Substituting formulae $(\bigstar)$, $(\ast)$ and $(\circledast)$ into formula $(\maltese)$ yields

$$\begin{aligned}\mathscr{P}(\theta_n) &= \varphi_n(X)\prod_{i=1}^{n}(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1}) \cap (\sigma_n \cdot X^{-n} \cdot (1 + X^{-1})^n) \\ &= \big((1+X)^n(1 + X^{q-1} - X^{q-2}z_i + \cdots - Xz_i^{q-2} + z_i^{q-1})\varphi_n(X)\big) \cap (\sigma_n \cdot X^{-n})\end{aligned}$$

from which the desired conclusion follows by [16] Theorem VI.6.2. □

Further interesting families of $\mathfrak{m}$-primary irreducible ideals are provided by starting with other sets of classical polynomials, such as the Dickson polynomials and/or varying the homogenization schema employed. It is not our intention to pursue this further here, but refer to [34], in particular the Appendix therein due to the second author, establishing a surprising connection between these matters and Steiner systems.

Appendix. A topological detail

In this Appendix we supply the missing topological details of the proof of Proposition 4.1. We preserve the notations of that proof. If $\zeta \downarrow B$ is a vector bundle of dimension ℓ then we denote by $1 + w_1(\zeta) + \cdots + w_\ell(\zeta) \in \mathrm{Tot}\big(H^*(B; \mathbb{F}_2)\big)$ its **total Stiefel–Whitney class**. For the closed surface $M = \underbrace{(S^1 \times S^1)\# \cdots \#(S^1 \times S^1)}_{t} \# \underbrace{\mathbb{RP}(2)\# \cdots \#\mathbb{RP}(2)}_{r}$ we need to show that any element $h = 1 + w_1 + w_2 \in \mathrm{Tot}\big(H^*(M; \mathbb{F}_2)\big)$, with $w_1 \in H^1(M; \mathbb{F}_2)$ and $w_2 \in H^2(M; \mathbb{F}_2)$, occurs as the total Stiefel–Whitney class of a 2-plane bundle over M.

Denote the connected sum components of M by $M_1, \ldots, M_s$ where $s = t + r$. Choose base points $\bullet \in M_i$ for $i = 1, \ldots, t$. A vector bundle of dimension d on M is determined by specifying vector bundles of dimension d on each of the components of the connected sum, using these to construct a vector bundle over the one point union $M_1 \vee \cdots \vee M_t$ of the components, and pulling the result back along the pinching map $M = M_1\# \cdots \#M_t \xrightarrow{q} M_1 \vee \cdots \vee M_t$. Similarly, cohomology classes $w_1 \in H^1(M; \mathbb{F}_2)$ and $w_2 \in H^2(M; \mathbb{F}_2)$ are determined uniquely by their restriction to the connected sum components. So we are reduced to showing the following: For any element $1 + w_1 + w_2$ with $w_1 \in H^1(\mathbb{T}; \mathbb{F}_2)$ and $w_2 \in H^2(\mathbb{T}; \mathbb{F}_2)$ or $w_1 \in H^1(\mathbb{RP}(2); \mathbb{F}_2)$ and $w_2 \in H^2(\mathbb{RP}(2); \mathbb{F}_2)$ there is a 2-plane bundle on $\mathbb{T}$ respectively on $\mathbb{RP}(2)$ with total Stiefel–Whitney class $1 + w_1 + w_2$.

We begin by describing the vector bundles we need over $\mathbb{T} = S^1 \times S^1$. Let $\lambda_L, \lambda_R \downarrow \mathbb{T}$ be the line bundles obtained from the canonical line bundle $\lambda \downarrow S^1$ by pulling it back along the projection onto the left, respectively right, S^1-factor of $\mathbb{T}$. Denote by $\xi \downarrow \mathbb{T}$ the 2-plane bundle obtained by pulling the tangent bundle $\tau_{S^2} \downarrow S^2$ back to $\mathbb{T}$ along the degree one map obtained by choosing a basepoint $\bullet \in S^1$ and collapsing the subset $S^1 \vee S^1 = (S^1 \times \bullet) \cup (\bullet \times S^1)$ of $S^1 \times S^1$ to a point.

Next, over $\mathbb{RP}(2)$ we need the canonical line bundle $\gamma \downarrow \mathbb{RP}(2)$ and the tangent bundle $\tau_{\mathbb{RP}(2)} \downarrow \mathbb{RP}(2)$. Recall that $(\tau_{\mathbb{RP}(2)} \oplus \mathbb{R}) \downarrow \mathbb{RP}(2) \cong (\gamma \oplus \gamma \oplus \gamma) \downarrow \mathbb{RP}(2)$ from which one can compute the Stiefel–Whitney classes of $(\tau_{\mathbb{RP}(2)} \oplus \mathbb{R}) \downarrow \mathbb{RP}(2)$ by the Whitney sum formula.[25]

Finally, write $H^*(\mathbb{T}; \mathbb{F}_2) = \mathbb{F}_2[x, y]\big/(x^2, y^2)$, where $x = w_1(\lambda_L)$ and $y = w_1(\lambda_R)$, and set $H^*(\mathbb{RP}(2); \mathbb{F}_2) = \mathbb{F}[u]\big/(u^3)$, where $u = w_1(\gamma)$. The following table shows that all the required possibilities are realized.

Table: Projective bundle threefolds.

Bundle	Total Stiefel–Whitney class	Bundle	Total Stiefel–Whitney class
$\mathbb{R}^2 \downarrow \mathbb{T}$	1	$\mathbb{R}^2 \downarrow \mathbb{RP}(2)$	1
$(\lambda_L \oplus \mathbb{R}) \downarrow \mathbb{T}$	$1 + x$	$(\gamma \oplus \mathbb{R}) \downarrow \mathbb{RP}(2)$	$1 + u$
$(\mathbb{R} \oplus \lambda_R) \downarrow \mathbb{T}$	$1 + y$	$(\gamma \oplus \gamma) \downarrow \mathbb{RP}(2)$	$1 + u^2$
$\big((\lambda_L \otimes \lambda_R) \oplus \mathbb{R}\big) \downarrow \mathbb{T}$	$1 + x + y$	$\tau_{\mathbb{RP}(2)} \downarrow \mathbb{RP}(2)$	$1 + u + u^2$
$\xi \downarrow \mathbb{T}$	$1 + x \cdot y$		
$(\xi \otimes \lambda_L) \downarrow \mathbb{T}$	$1 + x + x \cdot y$		
$(\xi \otimes \lambda_R) \downarrow \mathbb{T}$	$1 + y + x \cdot y$		
$(\lambda_L \oplus \lambda_R) \downarrow \mathbb{T}$	$1 + x + y + x \cdot y$		

25 This isomorphism also shows that the geometric dimension of $(\gamma \oplus \gamma \oplus \gamma) \downarrow \mathbb{RP}(2)$ is only two, not three, which is its actual dimension.

References

[1] J.F. Adams, On formulae of Thom and Wu, Proc. London Math. Soc.(3) 11 (1961) 741–752.
[2] J.F. Adams, C.W. Wilkerson, Finite H-spaces and algebras over the Steenrod algebra, Ann. of Math. 111 (1980) 95–143.
[3] J.F. Adams, C.W. Wilkerson, Finite H-spaces and algebras over the Steenrod algebra: a correction, Ann. of Math. 113 (1981) 621–622.
[4] W. Bruns, J. Herzog, Cohen–Macaulay Rings, in: Cambridge Studies in Advanced Math, vol. 39, Cambridge University Press, Cambridge, UK, 1993.
[5] H. Cartan, Algèbres d'Eilenberg–Mac Lane et Homotopie, Séminaire Henri Cartan, Ecole Normale Supérieure Paris, 7e année 1954/55, Secrétariat Mathématique, Paris 1956, W.A. Benjamin, New York 1967.
[6] W.G. Dwyer, C.W. Wilkerson, Poincaré duality and Steinberg's theorem on rings of coinvariants, Preprint, 2006.
[7] D. Glassbrenner, The Cohen–Macaulay property and f-rationality in certain rings of invariants, J. Algebra 176 (1995) 824–860.
[8] O.E. Glenn, Modular invariant processes, Bull. Amer. Math. Soc. 21 (1914-15) 167–173.
[9] H. Hasse, F.K. Schmidt, Noch eine Begrundung der Theorie des höheren Differentialquotienten in einem algebraischen Funktionkörper in einer Unbestimmten, J. Reine. Andgew. Math. 177 (1937) 215–237.
[10] K. Kuhnigk, Poincarédualitätsalgebren, Koinvarianten, und Wu-Klassen, Doktorarbeit, Universität Göttingen, 2003.
[11] K. Kuhnigk, On Macaulay duals of Hilbert ideals, J. Pure Appl. Algebra 210 (2007) 473–480. http://dx.doi.org.10.1016/jpaa.2006.10.14.
[12] T.-Z. Lin, Über Poincarédualitätsalgebra in der Invariantentheorie, Doktorarbeit, Universität Göttingen, Cuvillier Verlag, Göttingen, 2003.
[13] F.S. Macaulay, The algebraic theory of modular systems, in: Camb. Math. Lib, Cambridge University Press, Cambridge, UK, 1916, Reissued with an introduction by P. Roberts 1994.
[14] F.S. Macaulay, Modern algebra and polynomial ideals, Proc. Cambridge Philos. Soc. 30 (1934) 27–46.
[15] D.M. Meyer, L. Smith, Realization and nonrealization of Poincaré duality quotients of $\mathbb{F}_2[x, y]$ as topological spaces, Fund. Math. 177 (2003) 241–250.
[16] D.M. Meyer, L. Smith, Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations, in: Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2005.
[17] J.W. Milnor, The Steenrod algebra and its Dual, Ann. of Math. (2) 67 (1958) 150–171.
[18] S.A. Mitchell, Finite complexes with $\mathcal{A}(n)$ free cohomology, Topology 24 (1985) 227–248.
[19] M.D. Neusel, L. Smith, The Lasker–Noether theorem for $\mathcal{P}^*$-invariant ideals, Forum Math. 10 (1998) 1–18.
[20] O. Ore, On a special class of polynomials, Trans. Amer. Math. Soc. 35 (1933) 559–584.
[21] P. Schenzel, Über die freien Auflösungen extremaler Cohen–Macaulay Ringe, J. Algebra 64 (1980) 93–101.
[22] J.-P. Serre, Sur la dimension cohomologique des groupes profinis, Topology 3 (1964) 413–420.
[23] J.-P. Serre, Groupes finis d'automorphismes d'anneaux locaux réguliers, Colloq. d'Alg. Éc. Norm. Sup. de Jeunes Filles, Paris, 8-01–8-11, 1967.
[24] L. Smith, Polynomial Invariants of Finite Groups, A.K. Peters, Ltd., Wellesley, MA, 1995, second printing 1997.
[25] L. Smith, $\mathscr{P}^*$-invariant ideals in rings of invariants, Forum Math. 8 (1996) 319–342.
[26] L. Smith, Homological codimension of modular rings of invariants and the Koszul complex, J. of Math of Kyoto Univ. 38 (1998) 727–747.
[27] L. Smith, On a theorem of R. Steinberg on rings of coinvariants, Proc. Amer. Math. Soc. 131 (2003) 1043–1048.
[28] L. Smith, On alternating invariants and Hilbert ideals, J. Algebra 280 (2004) 488–499.
[29] L. Smith, An algebraic introduction to the Steenrod algebra, in: Proc. Intl. School and Conf. in Algebraic Topology Hà Nội 2004, in: Geometry & Topology Monograph, vol. 11, Math. Sci. Publ., 2007.
[30] L. Smith, On Steinberg's theorem on algebras of coinvariants, Forum Math. (2009) 965–979.
[31] L. Smith, R.E. Stong, About Macaulay inverses and Frobenius powers: a trip with Frobenius through Macaulay Country, Preprint, 2005.
[32] L. Smith, R.E. Stong, On $\mathfrak{m}$-primary ideals in $\mathbb{F}[x, y]$, Preprint, 2005.
[33] L. Smith, R.E. Stong, Invariants of binary forms modulo two, Proc. Amer. Math. Soc. 139 (2010) 17–26.
[34] L. Smith, R.E. Stong, Poincaré duality algebras mod Two, Adv. Math, (2010) in press (doi:10.1016/j.aim.2010.04.013).
[35] L. Smith, R.E. Stong, Coexact sequences of Poincaré duality algebras (in preparation).
[36] E.H. Spanier, Function spaces and duality, Ann 70 (1959) 338–378.
[37] E.H. Spanier, Algebraic Topology, McGraw Hill, New York, 1966.
[38] R.P. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1978) 57–83.
[39] R. Steinberg, Differential equations invariant under finite reflection groups, Trans. Amer. Math. Soc. 112 (1964) 392–400.
[40] N.E. Steenrod, Polynomial algebras over the algebra of cohomology operations, in: H-Spaces, Actes réunion Neuchâtel (Suisse), Auot 1970, in: Lecture Notes in Math., vol. 196, Springer-Verlag, Berlin, New York, 1971, pp. 85–99.
[41] N.E. Steenrod, D.B.A. Epstein, Cohomology Operations, in: Annals of Math. Studies, vol. 50, Princeton University Press, Princeton, 1962.
[42] R.E. Stong, Notes on Cobordism Theory, in: Math. Notes, Princeton Univ. Press, 1968.
[43] W.V. Vasconcelos, Ideals generated by R-sequences, J. Algebra 6 (1967) 309–316.
[44] J. Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, in: Commutative and Combinatorics, Kyoto 1985, in: Adv. Studies in Pure. Math., vol. 11, North–Holland, Amsterdam, 1987, pp. 303–312.
[45] R.M.W. Wood, An introduction to the Steenrod algebra through differential operators, Proc. Lond. Math. Soc. 3 (75) (1997) 194–220.
[46] R. Wood, Problems in the Steenrod algebra, Bull. Lond. Math. Soc. 30 (1998) 449–517.